On the Search for Stationary States in Quantum Mechanics

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Notion of the stationary quantum state is recalled. The classical way of finding stationary states for the Schrödinger equation is analysed. Attention is paid to the difference between the notion of physical stationary state and mathematical notion of function with separated variables. The mathematical basis of some general method for solving the Dirac equation is presented. The problem of finding stationary solutions for the case of the Dirac equation is analysed. A conclusion appears that there are no stationary states for the Dirac equation in a homogeneous electric field. Results of our paper [1] are confronted with some papers of other authors dealing with the problem of a homogeneous or constant electric field.

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1. Introduction

The message of this paper is to shed more light on the goals and motivation of our research paper [1] with an indication of the importance of the problem of finding stationary states for quantum equations. It can be treated as a kind of supplement to that paper.

Among the works on relativistic quantum mechanics, we found a few, upon reading which, one can have an impression that physicists have become so used to the existence of stationary states that they are convinced that there are stationary states in any electromagnetic field. Unfortunately, this is not true.

Perhaps this can be explained by the fact that a significant part of experimental research in quantum physics concerns atoms whose states are solutions of quantum equations for the Coulomb field, about which we know from experience that it has stationary states.

However, a mathematical proof of this fact in the case of the Dirac equation for the hydrogen atom is not trivial, and some of its mathematical details that are especially complex are omitted in the majority of quantum mechanics textbooks.

Moreover, some of these textbooks superficially present the method of finding stationary states, which may make physicists feel that this task is straightforward.

2. What is the stationary quantum state?

The history of the discovery of stationary states begins at the end of the 19th century. In recalling its basic facts we make use of [2].

Thanks to the intensive development of spectroscopy using prism and grid spectrometers, physicists observed that light emitted by elements in the gas phase had discontinuous linear spectrum.

According to the Thomson model of the atom, electron could emit light when was slightly displaced from its equilibrium position, at frequencies which should be equal to the frequency of the electron's vibrations or its harmonics. It remained mystery why the atoms did not emit light of frequencies that were harmonics of the fundamental frequency.

Rutherford's discovery of the nucleus enabled the very important hypothesis to be put forward that the electrons go round the nucleus in circular orbits, and the attractive Coulomb force is a centripetal force. However, according to the classical electrodynamics, an electron moving with an acceleration a, including centripetal one, radiates energy in the form of electromagnetic waves. The power radiated is $P = \frac{e^2 a^2}{6\pi\varepsilon_0 c^3}$W. Thus, an electron would lose energy and moving along spiral line after 10^{-8} s would fall into the nucleus.

As it appears from [3-5], Bohr soon after the appearance of Rutherford's model of the atom decided to improve it and successfully account for the hydrogen spectrum grouped into Balmer, Lyman, Paschen and other series.

To this end, he derived the formula for the energy of an electron in any circular orbit for a hydrogen atom and tried to match the difference of electron energies in two different orbits with the photon energies of experimentally observed spectral lines.

His effort was rewarded with success owing to his additional assumption that only some orbits were allowed. The total energy of the electron in such an orbit is precisely defined and constant over time. These orbits are called stationary.

When Bohr calculated the orbital angular momentum in such orbits, it turned out that it is equal to the multiple of the Planck constant. Due to its simplicity, it could be tempted to raise the significance of this fact to the rank of a postulate.

However, as an author points out [6], many physicists at that time considered such a postulate too incomprehensible to be treated directly as the foundation of quantum theory. They were convinced of the necessity of creating a new, more general theory that would replace classical mechanics in the field of atomic physics.

In the course of the search for such a theory, two directions of research emerged. The first one, indicated by Louis de 'Broglie, became known as wave mechanics, and the second one, indicated by Werner Heisenberg, was called matrix mechanics. Max Born coined the name of the new theory as quantum mechanics, and shortly thereafter Erwin Schrödinger demonstrated the equivalence of the both approaches to the problem.

In the course of the development of quantum mechanics, it became clear that the description of matter cannot consist only in assigning to a particle scalar values of mass, electric charge, energy or vector quantities of momentum or angular momentum.

It turned out that to such a physical object must be assigned a wave function describing the properties of matter in the time-space theater of physical events, and physical quantities are represented by operators that must act on the wave function in order to give the values of these physical quantities.

The wave function contains information not only about the values of physical quantities characteristic for a given particle, but also information about the direction of the course of a physical phenomena that a given particle is subject to.

Hence the need to emphasize emotionally (at least this is how we perceive it) that the wave function is not a mathematical function, but a physical

quantity of a special category, and therefore it has been given an alternative name of the state of a quantum particle.

Thus, particle states are divided into stationary ones, in which the particle has a constant energy over a time and does not interact with any quanta of some fields, and all other states, in which it interacts with quanta of these fields, i.e. non-stationary states.

It is obvious that in quantum mechanics stationary states or stationary wave functions should correspond to Bohr's stationary orbits in classical mechanics.

3. Searching for stationary states in a non-relativistic case

If we restrict our discussion to the case of electromagnetic interactions, then the description of the quantum particle comes down to solving the Schrödinger equation, as a result of that we obtain a countable or uncountable set of solutions, each of these is a single (scalar), generally complex mathematical function[1] dependent on the variables x, y, z, t.

The general form of the time-dependent Schrödinger's equation is

$$i\hbar\frac{\partial\Psi(\vec{r},t)}{\partial t} = \hat{H}\Psi(\vec{r},t), \tag{1}$$

where $\hat{H}$ is Hamiltonian operator of a system that normally contains a term describing the interaction of the system with an electromagnetic field.

Whether a given state Ψ is stationary[2] or not is determined by the behaviour of the wave function when the energy operator (2) acts on it

$$\hat{E} = i\hbar\frac{\partial}{\partial t}. \tag{2}$$

If as a result of action of this operator on a wave function we obtain an eigenvalue equation of the form

$$\hat{E}\Psi(\vec{r},t) = E\Psi(\vec{r},t), \tag{3}$$

[1] Usually marked with the symbol Ψ and called the wave function.

[2] For more discussions of properties of stationary states, see, for example, §8 of the book by Schiff [7], §16 by Davidov [8], §30 by Blokhintsev [9].

where E is a real number then the given wave function is called an eigenfunction of the operator with eigenvalue E and is said to represent a stationary[3] state of the system.

Standard textbooks on quantum mechanics [7-20] inform that in this case the time dependent part of the wave function should have the form

$$\chi(t) = e^{\frac{-iEt}{\hbar}}, \tag{4}$$

and the whole stationary state should be

$$\Psi(\vec{r}, t) = \psi(x, y, z)\chi(t). \tag{5}$$

As a matter of fact, searching for stationary states makes sense only if the Hamiltonian $\hat{H}$ of a system is a sum

$$\hat{H} = \hat{H}_0 + \hat{H}_I, \tag{6}$$

where the first term $\hat{H}_0$ does not depend on time (unperturbated Hamiltonian), and the second $\hat{H}_I$ describing an interaction of the system with the field that is in some sense small compared to the first term[4] (perturbation Hamiltonian). This means that solutions of the equation

$$i\hbar\frac{\partial\Psi(\vec{r}, t)}{\partial t} = \hat{H}_0\Psi(\vec{r}, t), \tag{7}$$

should be of the form (5), if (7) has stationary solutions.

So it would seem that now it is enough to insert (5) into (7) and solve the equation. Unfortunately, such carelessness can lead to wrong solutions.

A number of quantum mechanics textbooks [7-9,16,19-20] present full procedure of how to determine whether the solution of equation (7) can be written in the form (5), but there are also [10-15,17-18], that treat the problem somewhat superficially.

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[3] Such a definition of the stationary state is provided by Weyl [10], Davidov [8], Landau and Lifszyc [11], Messiah [12], while Schiff [7], Średniawa [20], Blokhintsev [9], Burkhardt and Leventhal [19], Sakurai [18], Merzbacher [17], Liboff [16] emphasize the importance of the independency of the probability density of finding the particle and all other probabilities from time.

[4] Moreover, the system described by H_0 should be conservative and the states of the whole system can be constructed by superposition of solutions of the unperturbated problem, Merzbacher [17].

We will show how this procedure should look like on the example of the two-dimensional wave equation. This part of our paper is based on the classical textbook of mathematical analysis [23].

The two-dimensional wave equation has the form

$$\frac{\partial^2 f(x,t)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 f(x,t)}{\partial t^2} = 0. \tag{8}$$

Let us omit here the boundary conditions of the equation as less important for the illustration of the problem. In order to determine whether (8) can have stationary solutions of the form (5), we must first examine whether solutions with separated variables t and x are allowed in this equation.

For this purpose[5], we propose as a trial solution

$$f(x,t) = T(t)X(x),$$

which we put in (8). As a result of the calculations, we get successively

$$T(t)\frac{\partial^2 X(x)}{\partial x^2} - \frac{1}{c^2}X(x)\frac{\partial^2 T(t)}{\partial t^2} = 0,$$

$$\frac{X^{"}(x)}{X(x)} = \frac{1}{c^2}\frac{T^{"}(t)}{T(t)}.$$

On the left side there is a function dependent only on the variable x, and on the right side only on t. Thus, equality is possible only if both the left side and the right side depend neither on x nor on t, i.e. they are equal to one and the same constant. Let us denote it as $-k^2$ and we have

$$\frac{X^{"}(x)}{X(x)} = \frac{1}{c^2}\frac{T^{"}(t)}{T(t)} = -k^2.$$

Hence, we obtain two independent equations to solve

$$X^{"}(x) + k^2 X(x) = 0,$$

$$T^{"}(t) + c^2 k^2 T(t) = 0.$$

[5] The Fourier method of solving partial differential equations presented here is only a way to determine whether it is permissible to search for solutions of a differential equation in the form $\psi(x,y,z)\omega(t)$. As the Schrödinger equation contains only one derivative over time and only of the first order, it is easy to equate the problem of separating a time variable from spatial variables with the problem of finding stationary states.

Only now solution of the second of these equations will allow us to answer the question whether equation (8) has stationary solutions. This procedure is general and applies to any partial differential equation.

As an illustration in the field of non-relativistic quantum mechanics, we propose the solution of the Schrödinger equation for the case of an electron moving in a homogeneous electric field presented in [11].

Assuming the field direction to be along the x axis, denoting as $F = eE$ the force acting on the particle in an electric field of intensity E, where the particle's charge is e [6], we have that the potential energy of the particle in this field is

$$U = -Fx + const.$$

After the authors of [11] we assume $const = 0$ so that for $x = 0$ it is $U = 0$. Thus the non-stationary Schrödinger equation in this field has the final form

$$i\hbar\frac{\partial\Psi(x,t)}{\partial t} = (-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - Fx)\Psi(x,t).$$

So let's substitute $\Psi(x,t) = T(t)X(x)$ and investigate whether in this case a solution with separated variables[7] is possible. When keeping standard calculations we get that

$$\frac{i\hbar\frac{\partial T(t)}{\partial t}}{T(t)} = \frac{-\frac{\hbar^2}{2m}\frac{\partial^2 X(x)}{\partial x^2}}{X(x)} - Fx = const.$$

Assuming $const = \epsilon$, a constant with the energy dimension, i.e. the energy of the stationary state[8], we can start solving independent equations for $T(t)$ and $X(x)$. After the authors [11] we quote that the solution with any energy ϵ is the following state

$$\Psi(x,t) = \frac{A}{\sqrt{\pi}}e^{\frac{-i\epsilon t}{\hbar}}\int_0^\infty cos(\frac{u^3}{3} - (x + \frac{\epsilon}{F})(\frac{2mF}{\hbar^2})^{\frac{1}{3}}u)du, \tag{9}$$

where A is a certain constant.

[6] The charge of the electron here, according to the notation of Landau and collaborators' Theoretical Physics course , is $e = -|e|$.

[7] The authors of [11] do not study that at all, this part of the calculations comes from us.

[8] Although the energy of the stationary state here is equal to ϵ , however, according to the kinetic energy operator $\hat{E} = i\hbar\frac{\partial}{\partial t} - eA_0$ for the particle in the field, the kinetic energy of the electron in this case is equal to $\epsilon + eEx$.

4. Searching for stationary states in a relativistic case

Again, we restrict our discussion to electromagnetic interactions case, and moreover to particles with spin $\frac{1}{2}$. This issue is described by the Dirac equation, which can be written as

$$i\hbar\frac{\partial\Psi(\vec{r},t)}{\partial t} = \hat{H}_D\Psi(\vec{r},t), \tag{10}$$

where $\hat{H}_D$ is the matrix of differential operators usually called the Dirac Hamiltonian. This notation is often used to emphasize the similarity of this equation to the Schrödinger equation.

However, the method of searching for stationary states in this case cannot be a carbon copy of this procedure for the Schrödinger equation, because the Dirac equation is a system of four first-order partial differential equations for a wave function being a system (a vector called a bispinor) of four mathematical complex scalar functions.

Typical university studies for physicists do not teach at all how to solve such mathematical problems. Moreover, contemporary textbooks dealing with partial differential equations, e.g. [24], rather focus on the evidence for the existence of weak or strong solutions to such problems.

Therefore, the best way out is to look through old textbooks on mathematical analysis that explain in detail these issues solved only in the 19th century. Just, we had three perfect textbooks printed in our native language [21, 22, 23].

After reading them, it can be concluded that the most promising course of action is to use the method that says that the integration of a system of differential equations can be reduced to the integration of one higher-order differential equation.

To make everything clear, we will present in detail the reasoning[9] included in [23]. For example, let us consider the case of a system of three first-

[9] We have slightly improved the description of this method contained in V. Smirnov's book to make it clearer, in our opinion. For this purpose, in particular, we have added the formula (20), which the author omitted in his text as probably too obvious. The presented method belongs to the classics of the theory of differential equations, because mathematical analysis textbooks do not indicate its author. Its description in the form of a theorem with the proof can also be found in §1 of chapter VII of the textbook [22]. As quantum mechanics textbooks are silent about this method, we have decided to provide here a detailed description of it.

order differential equations (for simplicity), solved with respect to the derivatives

$$\frac{dy_1}{dx} = f_1(x, y_1, y_2, y_3), \tag{11}$$

$$\frac{dy_2}{dx} = f_2(x, y_1, y_2, y_3), \tag{12}$$

$$\frac{dy_3}{dx} = f_3(x, y_1, y_2, y_3). \tag{13}$$

Let us assume that (11) does contain y_2 and we can solve it with respect to y_2. So we have

$$y_2 = \omega_1(x, y_1, \frac{dy_1}{dx}, y_3). \tag{14}$$

Formally calculating the derivative of the complex function y_2 (14) and substituting it for the left side (12) and substituting (14) for y_2 on the right side (12) we get a new version of equation (12)

$$\frac{\partial\omega_1}{\partial x} + \frac{\partial\omega_1}{\partial y_1}\frac{dy_1}{dx} + \frac{\partial\omega_1}{\partial(\frac{dy_1}{dx})}\frac{d^2y_1}{dx^2} + \frac{\partial\omega_1}{\partial y_3}\frac{dy_3}{dx} = \psi_2(x, y_1, \frac{dy_1}{dx}, y_3). \tag{15}$$

In turn substituting (14) to (13) we have a new version of that equation

$$\frac{dy_3}{dx} = \psi_3(x, y_1, \frac{dy_1}{dx}, y_3). \tag{16}$$

Finally, substituting ψ_3 from (16) to (15) for $\frac{dy_3}{dx}$ and solving (15) with respect to $\frac{d^2y_1}{dx^2}$ and marking ψ_3 as χ_3, we get a system of equations (17-18) equivalent to the original system (11-13)

$$\frac{d^2y_1}{dx^2} = \chi_1(x, y_1, \frac{dy_1}{dx}, y_3), \tag{17}$$

$$\frac{dy_3}{dx} = \chi_3(x, y_1, \frac{dy_1}{dx}, y_3). \tag{18}$$

Again assuming that we can solve equation (17) with respect to y_3 we can get from it that

$$y_3 = \varphi_3(x, y_1, \frac{dy_1}{dx}, \frac{d^2y_1}{dx^2}). \tag{19}$$

Formally calculating the derivative of y_3 as a complex function (19) and substituting it for the left side (18) and inserting (19) for y_3 to the right side (18) we get a new version of equation (18)

$$\frac{\partial\varphi_3}{\partial x} + \frac{\partial\varphi_3}{\partial y_1}\frac{dy_1}{dx} + \frac{\partial\varphi_3}{\partial(\frac{dy_1}{dx})}\frac{d^2y_1}{dx^2} + \frac{\partial\varphi_3}{\partial(\frac{d^2y_1}{dx^2})}\frac{d^3y_1}{dx^3} = \xi_1(x, y_1, \frac{dy_1}{dx}, \frac{d^2y_1}{dx^2}). \tag{20}$$

Equation (20) is simply a third order differential equation on y_1

$$\frac{d^3y_1}{dx^3} = F(x, y_1, \frac{dy_1}{dx}, \frac{d^2y_1}{dx^2}). \tag{21}$$

If we can integrate (21), its formal solution will be

$$y_1 = G(x, C_1, C_2, C_3). \tag{22}$$

Substituting (22) into (19) we get y_3 without any integrations, and substituting it into (14) together with (19) we get y_2 also without any integrations.

In the case of the Dirac equation, we therefore have in prospect replacing the system of four equations with a fourth-order equation, which is not pleasant news, but we must remember that solutions of the equation are complex functions.

Fortunately, there is spinor representation of the Dirac matrices [15], which allows us to simplify the problem by two orders of differentiation. Once more, we will deal with the problem of the homogeneous electric field, the source of which is only the scalar potential A_0. With such initial assumptions, the Dirac equation can be written in the form [1]

$$(\frac{\partial}{\partial x^0} + \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} - \frac{eA_0}{i\hbar c})\varphi(\overrightarrow{r}, t) = \frac{mc}{i\hbar}\chi(\overrightarrow{r}, t), \tag{23}$$

$$(\frac{\partial}{\partial x^0} - \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} - \frac{eA_0}{i\hbar c})\chi(\overrightarrow{r}, t) = \frac{mc}{i\hbar}\varphi(\overrightarrow{r}, t), \tag{24}$$

where $\varphi(\overrightarrow{r}, t)$ are two top components and $\chi(\overrightarrow{r}, t)$ are two bottom components of the bispinor wave function $\Psi(\vec{r}, t)$, $x_0 = ct$.

Since on the right-hand sides of (23-24) the spinors are multiplied by numbers, then using the above-described method of increasing the order of equations, we can replace (23-24) with a new system in the following form

$$(\boldsymbol{\nabla}^2 - \frac{\partial^2}{\partial(x^0)^2} + \frac{2eA_0}{i\hbar c}\frac{\partial}{\partial x^0} + \frac{e\boldsymbol{\sigma} \cdot \mathbf{E}}{i\hbar c} + \frac{e^2A_0^2}{\hbar^2c^2} - \frac{m^2c^2}{\hbar^2})\varphi(\overrightarrow{r}, t) = 0, \tag{25}$$

$$\chi(\overrightarrow{r}, t) = \frac{i\hbar}{mc}(\frac{\partial}{\partial x^0} + \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} - \frac{eA_0}{i\hbar c})\varphi(\overrightarrow{r}, t). \tag{26}$$

We actually have a second order (partial) differential equation for the upper bispinor, and the lower bispinor we can get from the upper one by differentiation only. However, the Pauli matrices in (25) mix components of the upper bispinor. In order to ultimately avoid solving the fourth order equation, we choose the direction of the electric field along the z axis, i.e. we assume

$$A_0 = \varepsilon z, \mathbf{E} \ = -\varepsilon \mathbf{k},$$

as only matrix σ_z is diagonal and where ε is a positive constant .

Since the particle now moves freely along the x and y axes, we can simplify the problem further by assuming that the momentum components of the particle along these axes are equal to zero. We then can get rid of differentiation over these variables in (25-26), and the spinors become functions of only variables z and x_0. Finally, we obtain a system of equations[10] equivalent to the original Dirac equation (23-24) in the following form

$$(\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial (x^0)^2} + \frac{2e\varepsilon z}{i\hbar c}\frac{\partial}{\partial x^0} - \frac{e\varepsilon\sigma_z}{i\hbar c} + \frac{e^2\varepsilon^2 z^2}{\hbar^2 c^2} - \frac{m^2c^2}{\hbar^2})\varphi(z,t) = 0, \qquad (27)$$

$$\chi(z,t) = \frac{i\hbar}{mc}(\frac{\partial}{\partial x^0} + \sigma_z\frac{\partial}{\partial z} - \frac{e\varepsilon z}{i\hbar c})\varphi(z,t). \qquad (28)$$

This way, solving the Dirac equation in a homogeneous electric field, the source of which is a scalar potential, comes down to solving one second-order differential equation, either on φ^1 (and χ^1 is obtained by differentiating the former) or on φ^2 (and χ^2 is obtained similarly to χ^1). There should be no doubt that components φ^1 and χ^1 are independent of φ^2 and χ^2, and each of them describes independent polarization states of the particle.

Before we examine whether equations (27-28) have stationary solutions, let us consider what the concept of stationarity means in the Dirac equation case?

Since the quantum state is described by a bispinor, then now each of its components should contain the factor given by (4) belonging to the same energy value E. Note that the above-described method of solving the Dirac equation facilitates the search for such solutions.

[10] It has not yet been specified in these equations whether we are investigating an electron or a positron.

Let us assume that the component φ^k $(k = 1, 2)$ already has the form (5). In order to obtain χ^k, according to (28), one should apply operators $\frac{\partial}{\partial x^0}$ and $\frac{\partial}{\partial z}$ to φ^k. However, regardless of making above differentiations the time dependence of bispinors χ^k will remain in the exponent of $e^{\frac{-iEt}{\hbar}}$. Thus, bispinors χ^k will also be of the form (5), although its dependence on spatial variable z may be different. We can see now that in order to determine whether the system of equations (27-28) has stationary solutions, it is enough to check whether equation (27) has them.

Let us now examine whether solutions φ^k $(k = 1, 2)$ can be stationary. Again, we first need to investigate whether we can separate equation (27).

Substituting $\varphi^k(z,t) = Z^k(z)T^k(t), (k = 1, 2)$ into (27) and making standard calculations we get

$$\frac{\frac{\partial^2 Z^k(z)}{\partial z^2}}{Z^k(z)} + \frac{e^2\varepsilon^2 z^2}{\hbar^2 c^2} + (-1)^k \frac{e\varepsilon}{i\hbar c} - \frac{m^2c^2}{\hbar^2} = \frac{(\frac{\partial^2}{\partial (x^0)^2} - \frac{2e\varepsilon z}{i\hbar c}\frac{\partial}{\partial x^0})T^k(t)}{T^k(t)}. \quad (29)$$

Due to the fact that on the right side of (29) exists the following term

$$-\frac{2e\varepsilon z}{i\hbar c}\frac{\partial}{\partial x^0}$$

mixing variables t and z, it is not possible to get separated solutions to the equation (27). Thus, the Dirac equation in a homogeneous electric field has no stationary solutions. Hence, there is nothing else but to find non-stationary solutions.

5. Conclussions

For many reasons, from around the 1940s, the development of relativistic quantum mechanics took place mainly in the field of quantum field theory, where perturbation-propagator methods based on Feynman diagrams reign supreme. Hence, at present, physics students are not taught systematic methods of solving the Dirac equation.

The few descriptions of such methods that are included in academic textbooks, for example in [15], did not satisfy us, among other things, because they were given substantially without proof. Hence, we decided to develop our own such method based on a clear mathematical justification.

The result of these activities is the method set out in the previous paragraph. The first problem we solved using it was the case of a homogeneous electric field presented in [1]. During making calculations contained in that paper, we did not know other attempts to solve this problem. Only after receiving its main results we decided to check whether there had been such attempts in the past. It turned out that there were several of them, and here is the effect of comparing the results of our work with the results of those works.

Fritz Sauter was the first who tried to solve the Dirac equation for the case of a uniform electric field [25]. He put the potential V into the form

$$V = \nu x$$

and looking for a solution made the following Ansatz

$$\psi = e^{\frac{i}{\hbar}(yp_y + zp_z - Et)}\chi(x). \tag{30}$$

Unfortunately, he did not investigate whether an electron had or not stationary wave functions in that field.

Next was Milton Plesset [26]. He considered the case for which V is a polynomial of any degree in x,

$$V = \sum_{n=0}^{q} a_n x^n, (0 < q < \infty).$$

He sought solutions of the same form as Sauter and also did not study the problem of existing stationary wave functions.

One can find another attempt made by Vernon Myers [27]. He assumed the solution to be

$$\Psi = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} e^{i\overrightarrow{k}\cdot\overrightarrow{r}},$$

where A_1, A_2, A_3, A_4 were functions of time , $k_x = k_{0x} + \frac{\epsilon E t}{\hbar}$, $k_y = k_{0y}$, $k_z = k_{0z}$, ϵ charge, E a constant, k_{0x} and k_{0y} and k_{0z} were constants. Components A_1, A_2, A_3, A_4 he obtained were rather complicated, given by a power series and exponent. They did not look like components of free bispinor.

And the authors of the last attempt we found were Fradkin, Gitman, and Shvartsman [28]. They studied the external electromagnetic field, which is a combination of parallel electric and magnetic fields given by potentials[11]

$$A_{\mu}^{ext}(x) = (0, -\kappa H\varphi(\frac{x_2}{\kappa}), 0, \alpha E g(\frac{x_0}{\alpha})),$$

where φ and g are arbitrary functions and κ and α are the dimensional control parameters. This potential gives the following field strengths

$$E_x = E_y = H_x = H_y = 0,$$

$$E_z = Eg^{'}, H_z = H\varphi^{'}.$$

To obtain a constant electric field it should be assumed that

$$g(\frac{x_0}{\alpha}) = \frac{x_0}{\alpha}.$$

To solve the Dirac equation, the authors [28] applied the general method given in §32 of the textbook [15], where is used the so-called quadratic Dirac equation and they postulated that its solution had the following form

$$\psi(x) = L^{-1}e^{-i(\frac{p_1x^1+p_3x^3}{\hbar})}f_{\lambda^2,\zeta}(x_2)\chi(x^0)u_{\zeta}, \tag{31}$$

where p_1 and p_3 are eigenvalues of operators $\mathrm{i}\hbar\partial_1$ and $\mathrm{i}\hbar\partial_3$, and u_{ζ} are eigenvalues of matrices α_3 and Σ_3

$$\Sigma_3 u_{\zeta} = \zeta u_{\zeta}, \alpha_3 u_{\zeta} = u_{\zeta}, \zeta = \pm 1.$$

Myers did not investigate whether the Dirac equation in a homogeneous electric field had stationary solutions. The solution he proposed was clearly non-stationary. However, there is no explanation in his work why this solution of such particular form should be correct. The form of his wave function is very sophisticated. This is why such form of solution should be well justified.

The other authors looked for solutions with separated variables, in addition the first two stationary ones. However, none of them investigated whether the analysed case of the Dirac equation allowed for solutions with separated variables.

[11] Here the notation is taken from [28].

We will pay more attention to the last example and quote the form of the quadratic Dirac equation that the authors [28] analysed. It has the following form

$$(P^2 - m^2c^2 - i\frac{e\hbar}{c}\alpha_3 Eg^{'} + \frac{e\hbar}{c}\Sigma_3 H\varphi^{'})\psi(x) = 0. \tag{32}$$

However, the term P^2 that the equation contains is a differential operator of the form [15]

$$-\hbar^2c^2\frac{\partial^2}{\partial {x_0}^2} - (i\hbar\frac{\partial}{\partial x_1} - \frac{e}{c}\kappa H\varphi(\frac{x_2}{\kappa}))^2 + \hbar^2\frac{\partial^2}{\partial {x_2}^2} - (i\hbar\frac{\partial}{\partial x_3} + \frac{e}{c}\alpha Eg(\frac{x_0}{\alpha}))^2.$$

As you can see, there are terms mixing the variables x_1 and x_2 as well as x_0 and x_3. Thus, at first glance, we can imagine the possibility of separating variables only in pairs, i.e.

$$\psi(x) = f(x_1, x_2)\chi(x_0, x_3),$$

and hence not completely.

As it results from [29], there are relatively simple methods giving solutions in which the temporal variable is separated from the spatial variables (the Fourier method for many independent variables), but the condition is that the coefficients of spatial derivatives can not be functions of time. In the case of equation (32), this is not the case. Hence, in our opinion, before using the substitution (31), a proof of the admissibility of such a solution should be provided.

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