Can quantum particle move along classical trajectory?

Janusz Szczączhor

johnsz@abs.umk.pl

Abstract: Similarities and differences between classical and quantum mechanics are recalled. The question of existing a classical trajectory to the quantum particle is asked. A wave function to the motion of a classical point is proposed. Original interpretation of a plane wave is given. The notion of wave function field is introduced.

PACS numbers:
03.65.Ta

1. Introduction

Although since the discovery of quantum mechanics 93 years had passed, still there are current many various, sometimes mutually contradictory, views on its interpretation [1,2]. The question why this still happens is obvious. In our opinion, to the question exists only one answer, i.e. the physical reality is more complicated than one could expect and the quantum physics can not be entirely squeezed into a Hilbert space.

Quantum and classical mechanics although are very different in their interpretations still have a lot in common. Leafing through physical literature one can find the following basic similarities between the two theories.

Similarity No. 1

Laws of classical mechanics expressed by the Poisson brackets formally turn into the laws of quantum mechanics when we replace these brackets with commutators divided by factor $i\hbar$, and functions $F(q,p,t)$ describing quantities in classical mechanics are represented by Hermitian linear operators $\hat{F} = F(\hat{q},\hat{p},t)$ describing the same quantities in quantum mechanics [3,4] (here q are positions i.e. $\mathbf{r}_l = \mathbf{r}_l(t)$, and p momenta i.e. $\mathbf{p}_l = \mathbf{p}_l(t)$, where the subscript l numbers particles, material points and so on).

Similarity No. 2

In the limit, where the Planck's constant approaches to zero the stationary Schrödinger equation for the wave function $\psi(q,t)$ reduces to the Hamilton - Jacobi equation for the action S of a particle with separated time where the phase of the wave function $\psi(q,t)$ multiplied by $\hbar$ turns into action $S(q,p)$ [5,6]. This is called the correspondence principle between quantum and classical mechanics.

Similarity No. 3

If we mean by the position and momentum vectors of the wave packet the weighted averages or expected values of these quantities then it can be shown that classical and quantum movements are always compatible with each other (the Ehrenfest's theorem)[4,6].

As for the differences, of course, the most serious are the following.

Difference No. 1

In classical mechanics [8,9,10] *the motion* of a particle is described by its radius vector $\mathbf{r}$. It gives its position as a function of time t

$$\mathbf{r} = \mathbf{r}(t). \tag{1}$$

Here we assume that the function $\mathbf{r}(t)$ is unique, continuous and differentiable at least twice. These assumptions matematically express the fact, among other things, that at a given moment of time t classical particle may exists only at one position $\mathbf{r}$.

Difference No. 2

In quantum mechanics [4] we assume that a wave function $\psi(\mathbf{r},t)$, which is a solution of appropriate wave equation, provides a quantum-mechanically complete description of the behavior of a particle of mass m with the potential energy $V(\mathbf{r},t)$ and hence is analogous to the classical trajectory (1). Further, it is assumed that the product of the wave function and its complex conjugate is the *position probability density*

$$P(\mathbf{r},t) = \psi^*(\mathbf{r},t)\psi(\mathbf{r},t) = |\psi(\mathbf{r},t)|^2. \tag{2}$$

This means that $P(\mathbf{r},t)dxdydz$ is the probability of finding a particle in its volume element $dxdydz$ about its point $\mathbf{r}$ at the time t. The probability of finding the particle somewhere in the region must be unity. Therefore, the wave function must be normalized in the entire area, at which it can be found, i.e.

$$\int |\psi(\mathbf{r},t)|^2 d^3r = 1. \tag{3}$$

Unfortunately, all these postulates to a certain extent reflect a desire to achieve unattainable excellence, since there are solutions of quantum equations that do not satisfy the condition (3), and still are physically correct.

A typical example of these is one of free solutions, for spin-up electron, of the Dirac equation in the standard representation [11]

$$\psi(\mathbf{r},t) = \begin{bmatrix} 1 \\ 0 \\ \frac{cp_z}{E+mc^2} \\ \frac{c(p_x+ip_y)}{E+mc^2} \end{bmatrix} e^{\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{r}-Et)}. \tag{4}$$

It can be normalized in every point $(\mathbf{r},t)$ in the sense that $\psi^\dagger\psi = 1$, but its norm

$$\int |\psi(\mathbf{r},t)|^2 d^3r \tag{5}$$

is infinite. In our opinion, the solution does not belong to a Hilbert space and $|\psi(\mathbf{r},t)|^2$ of (4) can not be treated as probability density. Similar situation occurs in non-relativistic case [12].

Given above, the question arises about physical meaning of expectation value of a dynamical variable, for example $\mathbf{r}$, when the system is in free state i.e.

$$\int \mathbf{r} \mid \psi(\mathbf{r},t) \mid^2 d^3r. \tag{6}$$

On the other hand, it seems to us that the solution to the above problem given by Dirac with introducing his delta function is actually a workaround, because this has been achieved through formal mathematical treatments. It is definitely a great tool for doing relativistic perturbation calculations in Collision Theory [11], which allows you to easily express the law of conservation of the 4-momentum. However, we believe it is only a way of expressing the orthonormality relation of continuous spectrum of the momentum operator, but it does not give us any physical interpretation of free wave function.

The fact that the state vectors (4) are difficult to interpret in statistical terms really weighs against the Heisenberg uncertainty principle, especially because, as Born spoke [13] 'a paper by Heisenberg containing his celebrated uncertainty principle, contributed more than the above-mentioned successes to the swift acceptance of the statistical interpretation of the ψ-function'. What is the matter then?

The proof of the Heisenberg uncertainty principle[1] is based on the assumption that the state vector of the system is normalized and there exist expected values of operators representing pairs of the physical quantities that are canonically conjugate to each other in the hamiltonian sense. Hence we get the conclusion that the Heisenberg uncertainty principle has not been proven to be correct when the system is described by a free wave function. Academic textbooks say about that nothing [3-6] and present some attempts to evade the problem with the help of various wave packets expanded in momentum eigenfunctions in order to show that the condition (3) is met by the wave packet. It is very disappointing.

2. Can quantum particle move along classical trajectory?

While we still were a student we had noticed that if the free electron could move along classical trajectory then thanks to the fact that such an object could exist in a given moment of time t only at one position $\mathbf{r}$(t) would disappear the problem of normalization of free wave function because there would be no sense to perform integrals (5) and (6). The fact that $|\psi(\mathbf{r})|^2$ is equal to 1 at every point of space would mean that the probability of finding a particle at the point where it is currently located is equal to 1, and at any other 0, not because $|\psi(\mathbf{r})|^2$ is equal to 0 , but due to the fact that particle possessing trajectory is not allowed to be there. Confirmation of this fact would obviously change wave function interpretation. But how to prove that?

As for the theory let us come back to the point 'Similarity No. 2' in the Introduction [2] and remind that the point of departure for deriving this property is the following wave function [5,6][3]

$$\psi(\mathbf{r}) = A(\mathbf{r})\, e^{\frac{i}{\hbar}S(\mathbf{r})}. \tag{7}$$

Comparing (7) to (4) we can see that in case of free motion $A(\mathbf{r}) = A = const.$. So, we shall study this similarity once again and this time we are starting from the free Dirac equation

$$i\hbar\frac{\partial\psi(\mathbf{r},t)}{\partial t} = \{c\,\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta mc^2\}\psi(\mathbf{r},t), \tag{8}$$

[1] From [14] it appears that currently recognized as a fully correct formulation of the uncertainty principle does not come from Heisenberg but from Kennard [15] and it is present in all quantum mechanics textbooks.

[2] In our opinion, Similarity No. 2 was discovered by Schrödinger in derivation of his equation in [7].

[3] The expression $e^{\frac{i}{\hbar}S(\mathbf{r})}$ for a wave function ψ was put in fact by Schrödinger in [7].

in the standard representation, that is,

$$\beta = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}, \boldsymbol{\alpha} = \begin{vmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{vmatrix}, \tag{9}$$

and as a trial wave function we take

$$\psi(\mathbf{r}, t) = \begin{vmatrix} \varphi \\ \chi \end{vmatrix} e^{\frac{i}{\hbar} S(\mathbf{r},\mathbf{t})}, \tag{10}$$

where φ and χ are independent of the coordinates two-component spin functions [4]. Assuming that the function S is of the form

$$S = \boldsymbol{p} \cdot \boldsymbol{r} - Et, \tag{11}$$

we insert (10) into (8), move all the terms (changing the sign, if necessary) to left-hand side and get

$$\begin{vmatrix} mc^2 + \frac{\partial S}{\partial t}, & c\boldsymbol{\sigma} \cdot \boldsymbol{\nabla} S \\ -c\boldsymbol{\sigma} \cdot \boldsymbol{\nabla} S, & mc^2 - \frac{\partial S}{\partial t} \end{vmatrix} \begin{vmatrix} \varphi \\ \chi \end{vmatrix} = 0. \tag{12}$$

It is a set of algebraic equations homogeneous in the unknowns and hence has solutions only if the determinant of the coefficients is zero. This is equivalent to the following request

$$(c\sqrt{(\boldsymbol{\nabla} S)^2 + m^2c^2} - \frac{\partial S}{\partial t})(c\sqrt{(\boldsymbol{\nabla} S)^2 + m^2c^2} + \frac{\partial S}{\partial t}) = 0. \tag{13}$$

This means that at least one of the factors of the product (13) is equal to 0. We can see that an expression in the second brackets is the relativistic Hamilton-Jacobi equation which is valid in classical mechanics

$$c\sqrt{(\boldsymbol{\nabla} S)^2 + m^2c^2} + \frac{\partial S}{\partial t} = 0. \tag{14}$$

The expression in the first parenthesis in classical mechanics is generally different from 0. Hence we can say that if is satisfied the free Dirac equation to the wave function (10), where S is given by formula (11) then is satisfied the relativistic Hamilton-Jacobi equation (14). Since the function (11) is the classical action of free particle then free electron (or positron) moves along classical trajectory.

As additional theoretical support for this conclusion one can quote [16], where is given a method of calculating the cross section for bound-free electron-positron pair production in a collision system consisting of a projectile ion colliding with a target ion. The authors claim that at relativistic

velocities, it is a very good approximation to assume a classical straight-line trajectory $\mathbf{R} = \mathbf{b} + \mathbf{v}t$ for the projectile motion, where $\mathbf{v}$ is the projectile velocity and $\mathbf{b}$ is the impact parameter.

Similar situation is in the case of homogeneous electric field. As was shown in [17] at this field the Dirac equation does not have any stationary solutions but only nonstationary ones. For spin up electron in the spinor representation the solution is as follows

$$\Psi_{+\frac{1}{2}}(z, x^0) = \begin{pmatrix} 1 \\ 0 \\ \frac{E-pc}{mc^2} \\ 0 \end{pmatrix} C_1 e^{i[-\frac{\omega^2}{2} ln|2(z+x^0)\sqrt{\alpha}+C|+\alpha(z^2+2zx^0-(x^0)^2)+C(z-x^0)\sqrt{\alpha}]}, \tag{15}$$

where E i p are functions given by

$$E_{+\frac{1}{2}} = \frac{m^2c^4}{2e\varepsilon}\frac{1}{z+ct+D} + \frac{e\varepsilon(z+ct+D)}{2}, \tag{16}$$

and

$$p_{+\frac{1}{2}} = -\frac{m^2c^3}{2e\varepsilon}\frac{1}{z+ct+D} + \frac{e\varepsilon(z+ct+D)}{2c}. \tag{17}$$

Here are used the following denotations

$$x^0 = ct,$$

$$\alpha = e\varepsilon/4\hbar c,$$

$$\omega = mc/2\hbar\sqrt{\alpha},$$

and ϵ is a positive constant describing the strength of the uniform electrostatic field

$$A_0 = \varepsilon z.$$

As can be seen from (16) and (17), quantum mechanics can not give here correct values of energy and momentum of accelerated particles. However, as was also shown in [17], it is sufficient to assume that the particle moves along classical trajectory according to the relativistic Newton's second law of motion and get correct values of the physical quantities. What is more, it turns out that contained in the exponent of formula (15) the following expression

$$-\frac{\omega^2}{2} ln \mid 2(z+x^0)\sqrt{\alpha} + C \mid +\alpha(z^2+2zx^0-(x^0)^2) + C(z-x^0)\sqrt{\alpha}, \tag{18}$$

when multiplied by the Planck's constant $\hbar$ and after substituting into it the classical trajectory becomes the classical action for a motion of an electron in this field.

As for the experimental aspect, the first that deserves special attention is J.J.Thomson's research on cathode rays [18].

According to [19], in his second method for getting the needed second relationship between q/m and v Thomson used a special tube, in which the cathode ray beam (i.e. electrons) could be sent through an electric field produced by some plates, in which region there could also be a magnetic field established by external coils, acting on the beam with force directed opposite to the force of the electric field. Any deflection of the beam could be measured by the scale S at the end of the tube.

The speed of the electrons was equal at least to $0.1c$. Based on this device one constructed later oscilloscope and TV set. In all of these devices motion of electrons is precisely controlled assuming that they move along classic trajectory and acts upon them classical Lorentz force

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \tag{19}$$

In practical studies of elementary particles one uses a number of other devices, the operation of which is also based on the assumption that an elementary particle moves along classical trajectory. One can especially point out the following [20].

1. Nuclear emulsion, where the track of a charged particle consists of clusters of silver atoms produced by it along its trajectory.

2. Cloud chamber, where a charged particle when passes through a supersaturated gas it ionize the gas and its molecules will condense on the ions and form a track.

3. Bubble chamber, where in liquid hydrogen at temperatures of about four degrees above absolute zero bubbles form on the ions left in the wake of a passing charged particle.

In the next section I will show how to construct a wave function for the classical point, starting with the fact that the probability of finding such a point is equal to *1* only at one position $\boldsymbol{r}(t)$ at a time t, and at any other *0* at the same time, and under what assumptions we obtain an expression identical to the wave function that is eigenfunction of the momentum operator in the coordinate representation.

3. The wave function of a classical particle

To begin with, let us assume that an unconstrained particle[4] with mass m is moving at a constant velocity v. So, the particle moves along a straight line. After all, for the time being let us do not focus on the position (or radius) vector of the particle, but try to describe its motion with the notion of probability.

All classical motions of a particle have one thing in common (including zero motion when the particle is at rest), i.e. at a given moment of time t the particle exists only at one position $\mathbf{r}(t)$. This means that the probability of finding the particle at the place where it is currently located is *1* and at any other is *0*.

This is important difference towards to the quantum image that is generated by typical stationary bound wave functions, eg. of the harmonic oscillator or hydrogen atom. In contrast to the classical image, those probability distributions are continuous. Therefore, here the use of probability density would lead to serious mathematical problems. That is why, we introduce the following function.

Definition 1. Let the function P states the existence of our particle. Its value is equal to 1 at the place where the particle is currently located, and at any other 0. It determines the probability of finding our particle at any place (but not the position probability density).

Owing to the above definition, in our view, there is no sense in evaluation an integral of this function over the whole space. Hence, the function must not be normalized, it is irrelevant.

As our point moves in a manner consistent with classical mechanics its motion can also be described by its position vector $\mathbf{r} = \mathbf{r}(t)$. Thus, for each position on its trajectory we have $P(\mathbf{r}(t)) = 1$, and at any others $P = 0$. Therefore, we can try to find the broadest possible class of functions with which we could describe the time evolution of our function P along the particle trajectory, at any subsequent moment of time t. It is also clear that the function P along the trajectory is continuous. [5]

[4] Here by a particle we mean a point-like object.

[5] We have a small problem because our function P is not continous at the positions where it changes its value from *1* to *0*. However, it will suffice if we get a formula that gives a non-zero value of P only along the path of our particle, and beyond we assume the function to be identically equal to 0.

In order to construct the function, we can use the following properties:

1) constancy of function P along the trajectory,

2) an initial value of the function at the moment of time t_0 , i.e. $P(\mathbf{r}(\mathrm{t}_0)) = 1$.

As for mathematical point of view, the function P is a scalar field, and the trajectory is an isotimic surface upon which the field has constant value equal to *1*.[6] According to that, we have [21,22,23]

$$dP = \boldsymbol{\nabla} P \cdot d\mathbf{r} = 0, \tag{20}$$

where $d\mathbf{r}$ is any incremental displacement entirely contained in this surface.

However, we are not interested in any movement along the trajectory, but only in those derived from the motion of our particle. According to the definition of instantaneous velocity we have

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = \mathbf{u}\frac{ds}{dt}, \tag{21}$$

where $\mathbf{u}$ - is a unit vector tangent to the trajectory that is given by the position vector $\mathbf{r}(t)$ and $ds = |d\mathbf{r}|$ (see e.g.[9]).

Therefore, on the basis of (21) we can write that

$$d\mathbf{r} = \mathbf{v}dt, \tag{22}$$

and

$$dP = \boldsymbol{\nabla} P \cdot \mathbf{v}dt = \boldsymbol{\nabla} P \cdot \mathbf{u}\, vdt = 0, \tag{23}$$

where v is

$$v = \frac{ds}{dt},$$

the speed of the particle, of course.

From formula (23), it appears that for classical motion of a particle at a constant velocity the gradient of scalar field P is always perpendicular to the unit vector $\mathbf{u}$, tangent to its path, regardless of the particle speed value.

It is also clear that for different unit vectors $\mathbf{u}$ (i.e. pointing out different directions in space) there are different gradient vectors perpendicular to them and also different scalar fields P.

[6] In these considerations we neglect the fact that this surface consists of points belonging to different instants of time.

For the first time, the time has come to refer to quantum mechanics because it should be noted that the functional form of the wave function describing the motion of a free quantum particle does not depend on both its momentum, and the vector $\mathbf{r}$, see (4).

In our view, the same result we will obtain in classical mechanics only when we assume that

$$\boldsymbol{\nabla} P = 0. \tag{24}$$

Scalar field P satisfying condition (24) always satisfies condition (23), irrespective of the direction and magnitude of the particle velocity vector. So we have just received the first limit on the functional form of the field P.

Please, do not be surprised at the survey of above versatility. It is an intentional striving to get closer to the universality of the eigenstate of momentum operator in quantum mechanics. By way of analogy we are looking for this type of universal form of wave function in classical mechanics.

In the case of our scalar field P that describes the classical motion along a straight line at a constant speed, condition (24) generates the following request.

Let the particle moves over a closed time interval $< t_0,\ t_1 >$. During the interval, it moves from the position $\mathbf{r}_0 = \mathbf{r}(\mathrm{t}_0)$ to another position $\mathbf{r}_1 = \mathbf{r}(\mathrm{t}_1)$. Initial value of the field P at the first position is

$$P(\mathbf{r}(t_0)) = 1. \tag{25}$$

Therefore, let us look for a possibly wide class of functional forms of field P satisfying the condition (25) and a conclusion resulting from (24), i.e.

$$\frac{\partial P(\mathbf{r}(t))}{\partial x^i} = 0, \tag{26}$$

where $i = 1, 2, 3$ for every position of the particle $\mathbf{r}(\mathrm{t})$ over the interval $< t_0,\ t_1 >$.[7]

There are many functional forms of field P meeting the upper conditions. The simplest is constant function everywhere equal to 1. But this is

[7] It should be stressed that the scalar field P is a function only of the particle rectangular coordinates x, y and z, and in no case the coordinates are intermediate arguments depending on time or the particle velocity.

very little interesting, because it does not contain any dynamics. Much more interesting is function P in the form of scalar product of two vector fields.

Of course, permitted class of such fields is again extremely broad, but we are interested, again by way of analogy with quantum mechanics, in a case of two two-dimensional vector fields giving P in the form of the following scalar product

$$P(\mathbf{r}) = g_{\alpha\beta}\Phi^{\alpha}(\mathbf{r})\Psi^{\beta}(\mathbf{r}), \tag{27}$$

where $g_{\alpha\beta}$ is metric tensor. Therefore, the condition (26) applied to the proposed field (27) takes on the following form

$$\begin{aligned}\frac{\partial P(\mathbf{r})}{\partial x^i} &= \frac{\partial}{\partial x^i}(g_{\alpha\beta}\Phi^{\alpha}(\mathbf{r})\Psi^{\beta}(\mathbf{r})) \\ &= g_{\alpha\beta}(\frac{\partial}{\partial x^i}\Phi^{\alpha}(\mathbf{r}))\Psi^{\beta}(\mathbf{r}) + g_{\alpha\beta}\Phi^{\alpha}(\mathbf{r})(\frac{\partial}{\partial x^i}\Psi^{\beta}(\mathbf{r})) = 0.\end{aligned} \tag{28}$$

About $g_{\alpha\beta}$ we assume that its components are constants. The question now arises, what dynamics we should give to the fields $\Phi^{\alpha}(\mathbf{r})$ and $\Psi^{\beta}(\mathbf{r})$?

We will not conceal the fact that we would like to obtain the fields as close as possible to the quantum ones, i.e.

$$\Phi^{\alpha}_{,i}(\mathbf{r}) = A^{\alpha}_{\sigma i}\Phi^{\sigma}(\mathbf{r}), \tag{29}$$

$$\Psi^{\beta}_{,i}(\mathbf{r}) = B^{\beta}_{\rho i}\Psi^{\rho}(\mathbf{r}), \tag{30}$$

where

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}. \tag{31}$$

We assume that A and B are matrices and its components are identical constants for each index $i = 1, 2, 3$.

Substituting (29) and (30) into (28) and changing "dummy" indices we obtain

$$(g_{\sigma\beta}A^{\sigma}_{\alpha i} + g_{\alpha\rho}B^{\rho}_{\beta i})\Phi^{\alpha}(\mathbf{r})\Psi^{\beta}(\mathbf{r}) = 0. \tag{32}$$

Again, we would like the condition (32) to be fulfilled in a manner independent of the particular form of fields $\Phi^{\alpha}(\mathbf{r})$ and $\Psi^{\beta}(\mathbf{r})$. The only such universal possibility is

$$A_i^T g + gB_i = 0, \tag{33}$$

written in a purely algebraic form, without indices [24].

Now, the only question is what form give to $g_{\alpha\beta}$? The analogy with the eigenstates of the momentum operator suggests to seek solutions of formal form

$$\Phi^{\alpha}(\mathbf{r}) = (\gamma(\mathbf{r}), \omega(\mathbf{r})), \tag{34}$$

$$\Psi^{\beta}(\mathbf{r}) = (\gamma(\mathbf{r}), -\omega(\mathbf{r})), \tag{35}$$

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{36}$$

where $\gamma(\mathbf{r})$ and $\omega(\mathbf{r})$ are new scalar functions continuous for variables x, y, z on the particle trajectory. Then the field P (27) has the form

$$P(\mathbf{r}) = \gamma^2(\mathbf{r}) + \omega^2(\mathbf{r}), \tag{37}$$

i.e. the square of the modulus of a complex wave function. Hence, condition (33) gives

$$a_1 = -b_1, \tag{38}$$

$$a_3 = b_2, \tag{39}$$

$$a_2 = b_3, \tag{40}$$

$$a_4 = -b_4. \tag{41}$$

Since the components of the matrices A and B are numbers, we can select some matrices which components are 0 or ± 1 as a basic vectors of the matrix space. There are many possible choices of sets of such vectors, but in our opinion the most interesting are the following

$$A_i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; B_i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{42}$$

To the finite displacement of our particle from the position indicated by the radius vector $\mathbf{r}_0$ to the next position indicated by $\mathbf{r_1}$ corresponds a finite transformation of field vectors $\Phi^{\alpha}(\mathbf{r})$ and $\Psi^{\beta}(\mathbf{r})$.

In order to obtain a form of the transformation let us expand any of the above fields about the point $\mathbf{r}_0$. Thus for any $\mathbf{r}_1$ ($\mathbf{r_1} = \mathbf{r}_0 + \Delta\mathbf{r}$) a Taylor's series expansion for field eg. $\Psi^{\beta}(\mathbf{r})$ gives [25]

$$\Psi^{\beta}(\mathbf{r}_0 + \Delta\mathbf{r}) = \sum_{n=0}^{\infty} \frac{1}{n!} (\Delta\mathbf{r} \cdot \boldsymbol{\nabla})^n \Psi^{\beta}(\mathbf{r}_0). \tag{43}$$

Furthermore, let us make the matrix B more specific (and analogically matrix A, too) as follows

$$B^{\rho}_{\beta i} = \frac{p_i}{\hbar}\bar{B^{\rho}_{\beta}}, \tag{44}$$

where

$$\bar{B^{\rho}_{\beta}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{45}$$

and p_i are components of a vector and $\hbar$ is a Planck's constant. Then the equation (30) will take the form

$$\Psi^{\beta}_{,i}(\mathbf{r}) = \frac{\partial}{\partial x^i}\Psi^{\beta}(\mathbf{r}) = \frac{p_i}{\hbar}\bar{B^{\beta}_{\rho}}\Psi^{\rho}(\mathbf{r}), \tag{46}$$

and the solution (43)

$$\Psi^{\beta}(\mathbf{r}_0 + \Delta\mathbf{r}) = \sum_{n=0}^{\infty}\frac{1}{n!}(\frac{\Delta\mathbf{r}\cdot\boldsymbol{p}}{\hbar})^n[(\bar{B})^n]^{\beta}_{\rho}\Psi^{\rho}(\mathbf{r}_0). \tag{47}$$

You can easily see that the evolution of the fields $\Phi^{\alpha}(\mathbf{r})$ and $\Psi^{\beta}(\mathbf{r})$ occurring while our point particle is moving from a point given by $\mathbf{r}_0$ to the another point given by vector $\mathbf{r}_0 + \Delta\mathbf{r}$ (where we mean $x = (\Delta\mathbf{r}\cdot\mathbf{p})/\hbar$) is given by [8]

$$\Phi(\mathbf{r}_0 + \Delta\mathbf{r}) = \begin{pmatrix} cosx & sinx \\ -sinx & cosx \end{pmatrix}\Phi(\mathbf{r}_0), \tag{48}$$

$$\Psi(\mathbf{r}_0 + \Delta\mathbf{r}) = \begin{pmatrix} cosx & -sinx \\ sinx & cosx \end{pmatrix}\Psi(\mathbf{r}_0). \tag{49}$$

The above calculations were deliberately carried out without the use of complex numbers to maintain a mathematical separation from quantum mechanics. Now we will show that our solutions are identical with the momentum operator eigenfunctions.

Let us replace fields $\Phi^{\alpha}(\mathbf{r})$ and $\Psi^{\beta}(\mathbf{r})$, formulae (34) and (35), with their complex versions according to the following scheme

$$\Phi^{\alpha}(\mathbf{r}) \rightarrow \Phi(\mathbf{r}) = \gamma(\mathbf{r}) + i\omega(\mathbf{r}), \tag{50}$$

$$\Psi^{\beta}(\mathbf{r}) \rightarrow \Psi(\mathbf{r}) = \gamma(\mathbf{r}) - i\omega(\mathbf{r}). \tag{51}$$

[8] We mention that there are many solutions to the equation (33), other than the given above, including particularly functions of the parameter x other than *sin* and *cos*, that is easy to achieve with another choice of matrices (42).

Of course that $\Phi(\mathbf{r}) = \Psi^*(\mathbf{r})$. Then equation (46) takes the form

$$\frac{\hbar}{i}\frac{\partial}{\partial x^k}\Psi(\mathbf{r}) = p_k\Psi(\mathbf{r}). \tag{52}$$

So function $\Psi(\mathbf{r})$ is an analogue of the wave function - an eigenstate of the momentum operator, and $\Phi(\mathbf{r})$ its complex conjugate. Equation (52) is the momentum operator eigenvalue equation. In this situation, the general solution (47) for the function $\Psi(\mathbf{r})$ takes the form

$$\Psi(\mathbf{r}_0 + \Delta\mathbf{r}) = e^{\frac{i\mathbf{p}\cdot\Delta\mathbf{r}}{\hbar}}\Psi(\mathbf{r}_0), \tag{53}$$

(as for $\Phi(\mathbf{r}_0 + \Delta\mathbf{r})$ it is enough to take its complex conjugate).

Let us now return to our function P($\mathbf{r}$) (27) representing the existence of our point particle. Using complex forms, (50) and (51), we can write that

$$P(\mathbf{r}) = \Psi(\mathbf{r})\Psi^*(\mathbf{r}). \tag{54}$$

If we choose as the initial condition, that

$$\gamma(\mathbf{r}_0) = 1, \tag{55}$$

and

$$\omega(\mathbf{r}_0) = 0, \tag{56}$$

then $P(\mathbf{r}_0) = 1$ and the general solution (53) implies that in every next position to which the body moves we will have

$$P(\mathbf{r}_0 + \Delta\mathbf{r}) = 1. \tag{57}$$

Let us now return to our assumptions about the classical motion of our point particle. Since the point is classical, and therefore it exists only at one position at a given moment of time, then it is impossible to sum the probabilities of finding it at different positions. The same, according to the assumptions, the function P states that at a given moment t the particle exists with a probability equal to 1 only at one place, and 0 at any other.

Let us emphasize, at which point the particle actually is placed points out the position vector, that is, its classical equation of motion. Therefore, the problem of normalizing of function $\Psi(\mathbf{r})$ naturally disappears.

Finally, note that we can put point $\mathbf{r}_0$ in point 0, i.e. in the origin of the coordinate system. Then vector $\Delta\mathbf{r}$ will be equal to the vector $\mathbf{r}$. So our solution (53) turns into $\Psi(\mathbf{r}) = \mathrm{e}^{\,i\mathbf{p}\cdot\mathbf{r}/\hbar}$, and its complex conjugate into $\Phi(\mathbf{r}) = \mathrm{e}^{-i\mathbf{p}\cdot\mathbf{r}/\hbar}$.

Resume

1) Wave function $\Psi(\mathbf{r}) = e^{i\mathbf{p}\cdot\mathbf{r}/\hbar}$ and its complex conjugate may represent the existence of a classical point particle moving along a straight line at a constant speed. However, the wave function does not give us full information about our particle. The main source of knowledge about it is still its position vector.

2) Our point particle states satisfy the momentum operator eigenvalue equation, and hence are eigenstates of the operator. Also from the above structure you can see that $\Phi(\mathbf{r})$ and $\Psi(\mathbf{r})$ are fields, those along with the motion of the point rotate by an angle depending on the momentum of the particle, the magnitude of the displacement $\mathbf{r}$, and the Planck constant.

At each position where the particle can be found, coexist both fields, both equally create the probability of finding the particle at that position.

As for the field $\Phi(\mathbf{r})$, it belongs to the same momentum eigenvalue $\mathbf{p}$, but by definition its argument is of the opposite sign.

4. Conclussions

This brief chapter starts with a reminder of facts not directly related to quantum mechanics.

Electrical and magnetic phenomena have been known for many hundreds of years. However, to Faraday's days one did not recognize their field character and also they were considered as physical phenomena independent of each other. Only this researcher pointed out their non-point and field character, and by discovering the phenomenon of electromagnetic induction showed that these phenomena are dependent on each other.

Today we know that electric and magnetic fields can transform into each other. They are relative quantities, i.e. their properties are different in different reference systems. In special cases, the electric field or magnetic field may be equal to zero in one reference system and at the same time be different from zero in another system. That is why with the passing of time, into physics the notion of electromagnetic field has been introduced.

In calculations it is represented by the electromagnetic field tensor

$$F_{ik} = \frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k},$$

where A_i are vector potential components.

However, experimentally measurable are only electric and magnetic fields, and therefore only they are physical quantities. The electromagnetic field can not be measured, it is only a physical construct, an ideal in that existence we firmly believe.

Into physics has been repeatedly introduced many such constructs. On the whole, we have grown accustomed to them so much that they do not excite us at the moment.

The prime example here is the notion of four-dimensional space-time, where it is usual to say that time is the fourth coordinate. This is not entirely true. First, because the fourth coordinate is really a variable

$$x^0 = ct. \tag{58}$$

Second, along the variables x, y, z we can move backwards and forwards.

When it comes to time, then for positive-energy particles it passes only forward, it is a parameter that determines the sequence of events. Thus, the time in formula (58) could be considered as true fourth dimension if the sequence of events would be parameterized by another variable, unrelated to x^0, but then we would live in the other world.

So, it should be naturally asked why we want the wave function to be physical quantity if it can not be experimentally measured?

In our view, as in the above two cases, the wave function, at this stage of the development of physics, is only a physical structure that does not need unequivocal physical interpretation. We should rather mean the wave function field, which consists of at least three types of wave functions related by the general idea of statistical interpretation, and differing in details of their individual interpretations.

Then, let us now suggest the existence of the following types of wave functions.

1. The first, that includes a wave function describing free motion of a quantum particle being the solution of either the Schrödinger or the Dirac equation, and a wave function describing the motion of a quantum Dirac particle in a homogeneous electric field derived from the scalar potential presented by us in [17].

At the same time, we disqualify in our view as a nonphysical but mathematically correct wave function, given by Schrödinger equation for this field [5].

These functions are interpreted as indicated above in Section 3, in particular at a given moment of time t particles of that type are present at a single position with probability equal to 1.

2. The second, which consists primarily of a wave function that is a sum of coherent, maybe originating from different starting points, but meeting at one point wave functions describing free motion of quantum particles which beams interfere with each other. This type also include functions describing diffraction and scattering processes.

   These functions can be clearly interpreted in statistical terms according to Born. About this type of wave function we will say that it comes into existence as a result of processing of free wave function.

   Particles of that type at a given moment of time t may be present at more than one position.

3. Finally, the third, that includes bound states of quantum particle which typical examples are electron states in hydrogen atom and states that describe the behavior of particles within the potential barrier provided that the potential barrier is higher than the particle energy.

   To these states we also attribute Born's statistical interpretation. However, wave function of this type differs from the above in that at least its one momentum component is imaginary, eg. the radial one.

   Particles of this type are also able to be present in more than one position at a given moment of time t, but in our view the physical mechanism of the possibility here may be different from that of the preceding type.

References

[1] L. E. Ballentine, Rev. Mod. Phys. **42**, 358(1970).

[2] J. G. Cramer, Rev. Mod. Phys. **58**, 647(1986).

[3] P. A. M. Dirac, The Principles of Quantum Mechanics, Pergamon, Oxford, 1958.

[4] L. Schiff, Quantum Mechanics, McGraw-Hill, New York, 1968.

[5] L.D. Landau, E.M. Lifshitz, Quantum Mechanics, Non-relativistic Theory, Pergamon, Oxford, 1958.

[6] A. Messiach, Quantum Mechanics, Volume I, North-Holland Publishing Company, Amsterdam 1967.

[7] E. Schrödinger, Quantisierung als Eigenwertproblem I, Annalen der Physik, **79**, 361 (1926).

[8] J. A. Shapiro, Classical Mechanics, http://www.physics.rutgers.edu/ugrad/494 bookr03D.pdf .

[9] W. Rubinowicz, W. Królikowski, Mechanika Teoretyczna, PWN, Warszawa, 1978.

[10] H. Goldstein, Ch. Poole, J. Safko, Classical Mechanics, Addison-Wesley, San Francisco, 2000.

[11] J. D. Bjorken, S. D. Drell, Relativistic Quantum Mechanics, McGraw-Hill, New York, 1964 .

[12] N. Lambert, Introductory Quantum Theory, http://www.mth.kcl.ac.uk ˜lambert/IntroQM.pdf .

[13] M. Born, The statistical interpretation of quantum mechanics, Nobel Lecture (1954).

[14] C. S. Calude, M. A. Stay, International Journal of Theoretical Physics **46**, 2013(2007).

[15] E. H. Kennard, Zeitschrift fur Physik **44**, 326(1927).

[16] J. Eichler, W. E. Meyerhof, Relativistic Atomic Collisions, Academic Press, San Diego, 1995.

[17] J. Szczązchor, On the wave function of relativistic electron moving in a uniform electric field, https://www.researchgate.net/, (2016).

[18] J. J. Thomson, Phil. Mag. **44**, 293, 311 (1897).

[19] D. L. Anderson, The Discovery of the Electron, D. Van Nostrand Company, Inc.,Princeton N. J., 1964.

[20] G. L. Wick, Elementary Particles, Geoffrey Chapman Publishers, London, 1972.

[21] E. Karaśkiewicz, Zarys Teorii Wektorów i Tensorów, PWN, Warszawa 1974.

[22] B. Hague, An Introduction to Vector Analysis, Methuen & Co. LTD., London, 1951.

[23] H. F. Davies, A. D. Snider, Introduction to Vector Analysis, Allyn and Bacon Inc., Boston, 1979.

[24] M. M. Postnikov, Lekcii po geometrii. Semestr 2, Linejnaja algebra, Nauka, Moskva, 1986.

[25] F. W. Byron, R. W. Fuller, Mathematics of Classical and Quantum Physics, Vol. 2, Addison-Wesley, Reading, 1970.