On the wave function of relativistic electron moving in a uniform electric field

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Solutions of the Dirac wave equation representing an electron moving in a uniform electric field are obtained. The spinor representation of $\boldsymbol{\alpha}$ and β matrices is applied. The wave functions are nonstationary. D'Alembert's method of solving second order partial differential equations is used. Nonexplicit expressions of energy and momentum are obtained. The expressions are relativistically correct. To obtain explicit values of them the quasi-classical interpretation of wave function is used. The probability of transmitting electrons through a uniform electric field barrier is calculated to be one.

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1. Introduction

Fritz Sauter was the first who tried to solve the Dirac equation for the case of a uniform electric field [1]. He put the potential V into the form

$$V = \nu x \tag{1}$$

and looking for a solution made the following Ansatz

$$\psi = e^{\frac{i}{\hbar}(yp_y + zp_z - Et)}\chi(x) \ . \tag{2}$$

Unfortunately, he did not investigate whether an electron had or not stationary wave functions in that field.

Next was Milton Plesset [2]. He considered the case for which V is a polynomial of any degree in x,

$$V = \sum_{n=0}^{q} a_n x^n, \ (0 < q < \infty). \tag{3}$$

He sought solutions of the same form as Sauter and also did not study the problem of existing stationary wave functions.

One can find another attempt made by Vernon Myers [3]. He treated the Dirac equation the same way as Sauter except that his solution was not stationary. He assumed the solution to be

$$\Psi = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{4}$$

where A_1, A_2, A_3, A_4 were functions of time , $k_x = k_{0x} + \frac{\epsilon E t}{\hbar}$, $k_y = k_{0y}$, $k_z = k_{0z}$, ϵ charge, E a constant, k_{0x} and k_{0y} and k_{0z} were constants. Components A_1, A_2, A_3, A_4 he obtained were rather complicated, given by a power series and exponent. They did not look like components of free bispinor.

All the above mentioned solutions have disadvanteges.

1. One can easily prove that non-stationary are all solutions of the Dirac equation for the motion of a charged particle in a uniform electrostatic field of infinite extent.
2. The uniform electric field is used in electrostatic accelerators where accelerated particles behave almost like the free ones. They easily pass through accelerating tube and are easily focused [4,5]. That is why one could expect a bispinor representing the Dirac particle moving in that field should resemble the free bispinor.

2. Stationary and non-stationary solutions of the Dirac equation

Dirac's equation may be put in the form of an expression for the time derivative [6,7]

$$i\hbar \frac{\partial \Psi(\mathbf{r},t)}{\partial t} = H\Psi(\mathbf{r},t), \tag{5}$$

where H is the Hamiltonian of the particle. The expression for H may be written as

$$H = \{c\,\boldsymbol{\alpha}\cdot(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}) + \beta mc^2 + eA^0\}, \tag{6}$$

where

$$\boldsymbol{p} = \frac{\hbar}{i}\boldsymbol{\nabla}, \tag{7}$$

is the momentum operator, $A^{\mu} = (A^0, \mathbf{A})$ is 4-vector potential of the electromagnetic field, Ψ is four-dimensional column vector (a bispinor), $\boldsymbol{\alpha}$ and β are Dirac matrices in the standard or in the spinor representation. The charge together with its sign is meant, so that for the electron $e = - \mid e \mid$.

When $A^0 \neq 0$ or $\mathbf{A} \neq 0$ the Dirac equation is a system of four partial differential equations. In order to find solutions of the Dirac equation for the motion of an electron in a uniform electrostatic field it is worth to make some choices as follows

1.
$$\mathbf{A} = 0, \quad A^0 = A_0 \neq 0, \tag{8}$$

2.
$$\Psi(\mathbf{r}, t) = \begin{pmatrix} \varphi(\mathbf{r}, t) \\ \chi(\mathbf{r}, t) \end{pmatrix}, \tag{9}$$

3.
$$\boldsymbol{\alpha} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & -\boldsymbol{\sigma} \end{pmatrix}, \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{10}$$

   where $\boldsymbol{\alpha}$ and β are Dirac matrices in the spinor representation and $\boldsymbol{\sigma}$ are Pauli matrices.

For subsequent reference Pauli matrices are written out below

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{11}$$

and for simplicity the notation is used

$$x^0 = x_0 = ct. \tag{12}$$

Hence the Dirac equation (5,6,7) splits up into two coupled equations and they can be written as

$$\left(\frac{\partial}{\partial x^0} + \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} - \frac{eA_0}{i\hbar c}\right)\varphi = \frac{mc}{i\hbar}\chi, \tag{13}$$

$$\left(\frac{\partial}{\partial x^0} - \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} - \frac{eA_0}{i\hbar c}\right)\chi = \frac{mc}{i\hbar}\varphi. \tag{14}$$

Now one multiplies equations (13) and (14) by $\frac{i\hbar}{mc}$ and introduces the notation

$$L^{+} = \frac{i\hbar}{mc}\left(\frac{\partial}{\partial x^0} + \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} - \frac{eA_0}{i\hbar c}\right), \tag{15}$$

$$L^{-} = \frac{i\hbar}{mc}(\frac{\partial}{\partial x^0} - \boldsymbol{\sigma}\cdot\boldsymbol{\nabla} - \frac{eA_0}{i\hbar c}), \tag{16}$$

so that (13) and (14) can be rewritten as

$$L^{+}\varphi = \chi, \tag{17}$$

$$L^{-}\chi = \varphi. \tag{18}$$

If one takes equation (17) as a formula for function χ and puts it into (18) just for χ, one will obtain an equation of the second order only for φ. One can treat likewise the function φ in equation (18) and insert it into (17) to receive an equation only for χ. Thus one obtains two systems of equations defining a solution of the Dirac equation, namely

$$L^{-}L^{+}\varphi = \varphi, L^{+}\varphi = \chi, \tag{19}$$

and

$$L^{+}L^{-}\chi = \chi, L^{-}\chi = \varphi. \tag{20}$$

The second order equations of both systems are independent of each other because, due to the fact that operators (15) and (16) do not commute, they are different equations. However, system of equations (19) as well as system (20) each of them individually are completely equivalent to the initial Dirac equation not only if the potential A^0 is non-zero, but also when $\mathbf{A}$ is, see [8]. Therefore, in order to get a full set of solutions of the Dirac equation one has to solve only one of these systems.

Now one would like to solve the second order equation of (19) that may be written as

$$(\boldsymbol{\nabla}^2 - \frac{\partial^2}{\partial(x^0)^2} + \frac{2eA_0}{i\hbar c}\frac{\partial}{\partial x^0} + \frac{e\boldsymbol{\sigma}\cdot\mathbf{E}}{i\hbar c} + \frac{e^2A_0^2}{\hbar^2c^2} - \frac{m^2c^2}{\hbar^2})\varphi = 0. \tag{21}$$

Still one can simplify (21) without any loss of generality and that is why one takes the uniform electrostatic field to be in the negative z direction

$$A_0 = \varepsilon z, \mathbf{E} = -\varepsilon\mathbf{k}, \tag{22}$$

where ε is a positive constant, and takes momentum components p_x and p_y to be zero. Therefore, spinor φ is a function of z and x^0 alone.

The fundamental question is whether or not equation (21) has stationary solutions? Setting

$$\varphi(z,t) = Z(z)T(t), \tag{23}$$

and separating the variables, equation (21) takes the form,

$$\frac{\frac{\partial^2 Z(z)}{\partial (z)^2}}{Z(z)} + \frac{e^2\varepsilon^2 z^2}{\hbar^2 c^2} - \frac{e\varepsilon\sigma_z}{i\hbar c} - \frac{m^2c^2}{\hbar^2} = \frac{(\frac{\partial^2}{\partial (x^0)^2} - \frac{2e\varepsilon z}{i\hbar c}\frac{\partial}{\partial x^0})T(t)}{T(t)}. \tag{24}$$

Because of the term

$$-\frac{2e\varepsilon z}{i\hbar c}\frac{\partial}{\partial x^0}, \tag{25}$$

that depends on both variables x^0 and z, on the right-hand side of (24) one can not have a function of only t set equal to a function of only z . This leads us to a conclusion that a Dirac particle, electron or positron or miuon, moving in a uniform electrostatic field does not have any stationary wave functions.

3. Search for d'Alembert's solution of the Dirac equation

The time has come to take into consideration one of leptons, namely an electron and set $e \to -e$. Taking into account (22) one obtains from (21) the following equation for spinor φ

$$(\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial (x^0)^2} - \frac{2e\varepsilon z}{i\hbar c}\frac{\partial}{\partial x^0} + \frac{e\varepsilon\sigma_z}{i\hbar c} + \frac{e^2\varepsilon^2 z^2}{\hbar^2 c^2} - \frac{m^2c^2}{\hbar^2})\varphi = 0. \tag{26}$$

From (19) the equation for spinor χ then reads

$$\chi = \frac{i\hbar}{mc}(\frac{\partial}{\partial x^0} + \sigma_z\frac{\partial}{\partial z} + \frac{e\varepsilon z}{i\hbar c})\varphi. \tag{27}$$

The second order equation (26) in two variables z and x^0 is hyperbolic [9,10,11]. Typical hyperbolic equation is the wave equation. It and the other hyperbolic equations can be solved by the method of characteristics. The solution of these equations is usually known as *d'Alembert's solution.* One route to the solution begins with a change of variables. Let

$$x = (z - x^0)\sqrt{\alpha}, \tag{28}$$

$$y = (z + x^0)\sqrt{\alpha} \tag{29}$$

be new co-ordinates and

$$\alpha = e\varepsilon/4\hbar c \tag{30}$$

$$\omega = mc/2\hbar\sqrt{\alpha}. \tag{31}$$

new constants. Of course spinors φ and χ are

$$\varphi = \begin{pmatrix} \varphi^1 \\ \varphi^2 \end{pmatrix}, \chi = \begin{pmatrix} \chi^1 \\ \chi^2 \end{pmatrix}. \tag{32}$$

Owing to the fact (11) that σ_z matrix is diagonal the upper component φ^1 is independent of the lower φ^2 and also the upper component χ^1 is independent of the lower component χ^2. Finally, one is ready to find φ^1 and χ^1. In order to do that one has to solve the following set of equations

$$(\frac{\partial^2}{\partial x \partial y} - i(x+y)(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}) + (x+y)^2 - i - \omega^2)\varphi^1 = 0, \tag{33}$$

$$\chi^1 = \frac{i}{\omega}(\frac{\partial}{\partial y} - i(x+y))\varphi^1. \tag{34}$$

Let us focus on equation (33). There is Riemann's method for solving any linear hyperbolic partial differential equations of the second order in two independent variables but the solution is too complicated to use for practical application. That is why we will use an idea of trial function from the Ritz variational method [12]. One can observe that a function

$$\varphi^1 = exp[i\{(x+y)^2/2\}] \tag{35}$$

almost satisfies equation (33). Therefore, one takes trial function of the form

$$\varphi^1 = exp[i\{(x+y)^2/2 + bf(x,y)\}], \tag{36}$$

where b is a constant and substitutes it into (33). It is expected to obtain a set of equations for function *f(x,y)* that is easy to integrate.

After some differentations and simplifications one obtains an equation for φ^1

$$[b(i\frac{\partial^2 f}{\partial x \partial y} - 2(x+y)\frac{\partial f}{\partial y} - b\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}) - \omega^2]\varphi^1 = 0. \tag{37}$$

It is satisfied only when the expression in square brackets

$$b(i\frac{\partial^2 f}{\partial x \partial y} - 2(x+y)\frac{\partial f}{\partial y} - b\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}) - \omega^2 \tag{38}$$

is equal to zero. It is complex functional relation and is equal to zero only when its real part is equal to zero and its imaginary part, too. The imaginary part of (38) equated to zero reads

$$\frac{\partial^2 f}{\partial x \partial y} = 0. \tag{39}$$

It is the wave equation in new variables x and y. The solution to it is

$$f(x,y) = g(x) + h(y). \tag{40}$$

This leads us to the conclusion that a solution of the equation (39) is a sum of a function g of x alone and a function h of y alone.

The real part of (38) equated to zero gives

$$b\frac{dh}{dy}(2y + 2x + b\frac{dg}{dx}) + \omega^2 = 0. \tag{41}$$

Since function g must not depend on variable y then the equation for the function must take the following form

$$2x + b\frac{dg}{dx} = C. \tag{42}$$

On the other hand function h must not depend on variable x and equation for it must take the form

$$(2y + C)b\frac{dh}{dy} + \omega^2 = 0. \tag{43}$$

C is a constant. Both equations (42) and (43) are easy to integrate and their solutions are

$$g(x) = \frac{-x^2 + Cx + C_1}{b}, \tag{44}$$

$$h(y) = \frac{-\omega^2}{2b} ln|2y + C| + C_2. \tag{45}$$

C_1 and C_2 are some new constants. Since *g(x)* and *h(y)* are already known then spinor component φ^1 is

$$\varphi^1 = C_1 e^{i[-\frac{\omega^2}{2} ln|2y+C| + \frac{(x^2+y^2)}{2} - x^2 + Cx]}, \tag{46}$$

where C and C_1 are some constants. In order to obtain χ^1 one differentiates φ^1 with respect to y and from (34) it follows that

$$\chi^1 = \frac{\omega}{2y + C}\,\varphi^1. \tag{47}$$

Finally, a wave function that is the nonstationary solution of the set of equations (33) and (34) can be expressed in the form:

$$\Psi(x, y) = \begin{pmatrix} 1 \\ 0 \\ \frac{\omega}{2y+C} \\ 0 \end{pmatrix} C_1 e^{i[-\frac{\omega^2}{2} ln|2y+C| + \frac{(x^2+y^2)}{2} - x^2 + Cx]}. \tag{48}$$

4. General properties of the nonstationary solution of the Dirac equation

Before investigating properties of the resulting electron wave function (48) it will be better if one goes back to the old variables. The state vector in variables z and x^0, according to formulae (28) and (29), takes the following form

$$\Psi(z,x^0) = \begin{pmatrix} 1 \\ 0 \\ \frac{mc^2}{e\varepsilon}\frac{1}{z+x^0+D} \\ 0 \end{pmatrix} C_1 e^{i[-\frac{\omega^2}{2}ln|2(z+x^0)\sqrt{\alpha}+C|+\alpha(z^2+2zx^0-(x^0)^2)+C(z-x^0)\sqrt{\alpha}]}, \tag{49}$$

where new constant D has been introduced

$$D = C/2\sqrt{\alpha}\ . \tag{50}$$

It is time to get to know the energy and momentum of an electron in a uniform electric field. According to [6] the kinetic momentum operator is

$$q^\mu = i\hbar\frac{\partial}{\partial x_\mu} - eA^\mu. \tag{51}$$

Now arises the question whether the state vector (49) is an eigenfunction of energy or momentum operators?

Through the action of operator q^0 (51) on the function $\Psi(z,x^0)$ (49) one obtains

$$q^0\Psi(z,x^0) = E_{+\frac{1}{2}}\Psi(z,x^0) + \tag{52}$$

$$+\begin{pmatrix} 0 \\ 0 \\ \frac{mc^2}{e\varepsilon}\frac{-i\hbar c}{(z+x^0+D)^2} \\ 0 \end{pmatrix} C_1 e^{i[-\frac{\omega^2}{2}ln|2(z+x^0)\sqrt{\alpha}+C|+\alpha(z^2+2zx^0-(x^0)^2)+C(z-x^0)\sqrt{\alpha}]},$$

where

$$E_{+\frac{1}{2}} = \frac{m^2c^4}{2e\varepsilon}\frac{1}{z+ct+D} + \frac{e\varepsilon(z+ct+D)}{2}. \tag{53}$$

Respectively, operator q^3 (51) acting on the same function (49) yields

$$q^3\Psi(z,x^0) = p_{+\frac{1}{2}}\Psi(z,x^0) + \tag{54}$$

$$+\begin{pmatrix} 0 \\ 0 \\ \frac{mc^2}{e\varepsilon}\frac{i\hbar}{(z+x^0+D)^2} \\ 0 \end{pmatrix} C_1 e^{i[-\frac{\omega^2}{2}ln|2(z+x^0)\sqrt{\alpha}+C|+\alpha(z^2+2zx^0-(x^0)^2)+C(z-x^0)\sqrt{\alpha}]},$$

where

$$p_{+\frac{1}{2}} = -\frac{m^2c^3}{2e\varepsilon}\frac{1}{z+ct+D} + \frac{e\varepsilon(z+ct+D)}{2c}. \tag{55}$$

Subscript $+\frac{1}{2}$ denotes spin-up state.

If imaginary coefficients $\frac{mc^2}{e\varepsilon}\frac{-i\hbar c}{(z+x^0+D)^2}$ in (52) and $\frac{mc^2}{e\varepsilon}\frac{i\hbar}{(z+x^0+D)^2}$ in (54) were equal to zero, $\Psi(z,x^0)$ would be an eigenfunction of energy and momentum operators.

However, the question comes to one's mind whether in the quantum mechanics the linear eigenvalue equations associated with energy and momentum operators are correctly formulated?

It is worthy noticing that one of the postulates of quantum mechanics tells us that the eigenvalues of all operators that represent physically measurable quantities are real numbers. On the other hand, pointing out many convincing arguments, in [16] Schiff states that a wave function that represents a particle traveling in the positive z-direction with precisely known momentum p and kinetic energy E should be harmonic function of function

$$\frac{i}{\hbar}(pz - Et), \tag{56}$$

and, at last for a free particle, the energy and momentum can be represented by differential operators that act on the wave function.

Taking into account that an electron moving in a uniform electric field behaves in a manner that is similar to that of free particle one comes to a conclusion that differentiation of exponential function in (49) is the only 'source' of electron energy and momentum, because only the function is harmonic. In the process, column elements of bispinor in (49) do not give any contribution to the energy or the momentum. That is why $E_{+\frac{1}{2}}$ in (53) is the energy eigenvalue and $p_{+\frac{1}{2}}$ in (55) is the momentum eigenvalue. Thus, energy and momentum eigenvalue equations should be redefined.

In [17] Nikishov and Ritus encountered the same problem, proceeded the same way but gave no explanation. Therefore, we felt obliged to present the above explanation.

One can easily observe that energy (53) and momentum (55) fulfil the following statement

$$\frac{E - pc}{mc^2} = \frac{mc^2}{e\varepsilon}\frac{1}{z+ct+D} \tag{57}$$

and owing to this the wave function (49) takes the final form as follows

$$\Psi_{+\frac{1}{2}}(z,x^0)=\begin{pmatrix}1\\0\\\frac{E-pc}{mc^2}\\0\end{pmatrix}C_1e^{i[-\frac{\omega^2}{2}ln|2(z+x^0)\sqrt{\alpha}+C|+\alpha(z^2+2zx^0-(x^0)^2)+C(z-x^0)\sqrt{\alpha}]}. \tag{58}$$

By direct substitution, you can check that received eigenvalues of energy (53) and momentum (55) satisfy familiar relationship (59), which confirms our belief that above conclusion was fully justified. (Below if they are not necessary the subscripts $+\frac{1}{2}$ or $-\frac{1}{2}$ will be omitted.)

$$E^2=p^2c^2+m^2c^4. \tag{59}$$

As regards the bispinor that represents the spin-down state, it is given by a solution of the following set of equations

$$(\frac{\partial^2}{\partial x\partial y}-i(x+y)(\frac{\partial}{\partial x}-\frac{\partial}{\partial y})+(x+y)^2+i-\omega^2)\varphi^2=0, \tag{60}$$

and

$$\chi^2=\frac{i}{\omega}(-\frac{\partial}{\partial x}-i(x+y))\varphi^2. \tag{61}$$

The solution is

$$\Psi_{-\frac{1}{2}}(z,x^0)=\begin{pmatrix}0\\1\\0\\\frac{E+pc}{mc^2}\end{pmatrix}C_1e^{-i[-\frac{\omega^2}{2}ln|2(z-x^0)\sqrt{\alpha}+C|+\alpha(z^2-2zx^0-(x^0)^2)+C(z+x^0)\sqrt{\alpha}]}, \tag{62}$$

where eigenvalues of energy and momentum are

$$E_{-\frac{1}{2}}=\frac{m^2c^4}{2e\varepsilon}\frac{1}{z-ct+D}+\frac{e\varepsilon(z-ct+D)}{2}, \tag{63}$$

and

$$p_{-\frac{1}{2}}=\frac{m^2c^3}{2e\varepsilon}\frac{1}{z-ct+D}-\frac{e\varepsilon(z-ct+D)}{2c} \tag{64}$$

respectively. The statement (59) also holds.

Both formulae (53) and (63) can be written as

$$E_{\pm\frac{1}{2}}=\frac{m^2c^4}{2e\varepsilon}\frac{1}{z\pm ct+D}+\frac{e\varepsilon(z\pm ct+D)}{2}. \tag{65}$$

In order to make a graph of function $E_{\pm\frac{1}{2}}$ for a moment we introduce new variable $x = e\varepsilon(z \pm ct + D)$ measured in MeV. Then for electron $E_{\pm\frac{1}{2}}(x) = \frac{(0{,}511)^2}{2x} + \frac{x}{2}$ and is given in Figure 1.

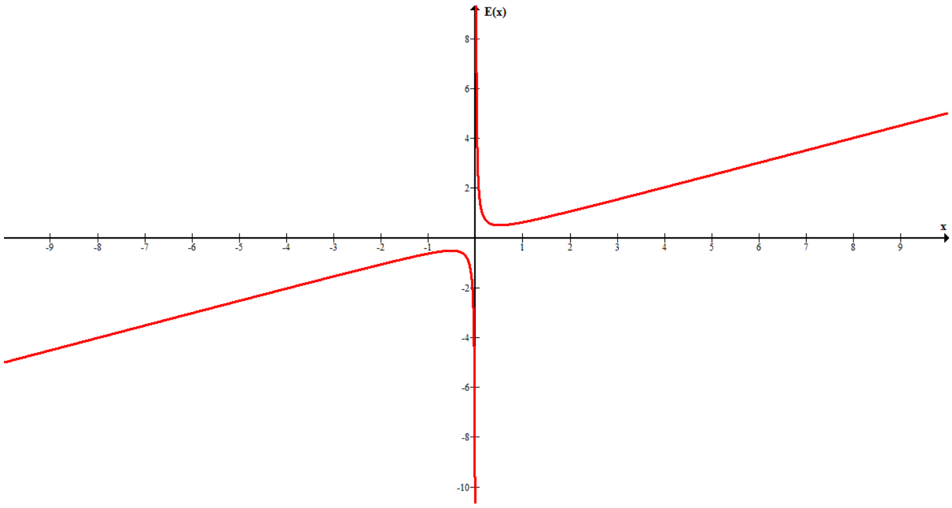

Figure 1: Plot of $E_{\pm\frac{1}{2}}(x)$

Function $E_{\pm\frac{1}{2}}(x)$ is not continous only at $x = 0$. One can easily see that if x is negative the values of $E_{\pm\frac{1}{2}}(x)$ are negative, too. But when x is positive the values of $E_{\pm\frac{1}{2}}(x)$ are also positive.

Let us introduce the following notation

$$u = z \pm ct + D, \tag{66}$$

$$a = m^2c^4/2e\varepsilon, \tag{67}$$

$$b = e\varepsilon/2. \tag{68}$$

Equation (65) now might be rewritten as

$$E(u) = \frac{a}{u} + bu. \tag{69}$$

One knows that E(u) has a minimum or a maximum at u_0 if $E'(u)$ exists and $E'(u_0) = 0$. We differentiate $E(u)$ and the derivative equate to 0

$$\frac{dE(u)}{du} = -\frac{a}{u^2} + b = 0. \tag{70}$$

The solution is $u = \pm\frac{mc^2}{e\varepsilon}$ and the value we denote as $\pm\delta$. The second-order derivative of $E(u)$ is

$$\frac{d^2E(u)}{du^2} = \frac{2a}{u^3}. \tag{71}$$

At $u = -\delta$ the second-order derivative is negative, hence $E(u)$ has local maximum equal to

$$E(-mc^2/e\varepsilon) = -mc^2, \tag{72}$$

and if $u = +\delta$ then the second-order derivative is positive and $E(u)$ has local minimum equal to

$$E(mc^2/e\varepsilon) = mc^2. \tag{73}$$

From formulae (55) and (64) it follows that at $u = \pm\frac{mc^2}{e\varepsilon}$ electron momentum is equal to 0 and at others u the momentum has always non-zero values.

We conclude, therefore, an electron moving in a uniform electric field has exactly the same set of possible values of energy and momentum as the free electron. When the variable u is negative $E(u)$ and $p(u)$ describe negative energy solutions of the Dirac equation but when u is positive they describe the positive ones. The same wave function represents negative and positive energy solutions.

Finally, one ought to consider wave functions (58) and (62) in the context of statistical interpretation of quantum mechanics. Unfortunately, things do not look too good.

As a model probability density we can use $\Psi^{\dagger}_{+\frac{1}{2}}(z,x^0)\Psi_{+\frac{1}{2}}(z,x^0)$ equal to

$$\varrho(z,x^0) = C_1^2\Big(1 + \frac{m^2c^4}{e^2\varepsilon^2}\frac{1}{(z+x^0+D)^2}\Big). \tag{74}$$

Let us take into account the following integral

$$\int \varrho(z,x^0)dz \tag{75}$$

and for the moment x^0 be an arbitrary but fixed point in time. Since function

$$\frac{1}{(z+x^0+D)^2} \tag{76}$$

in (74) is always non-negative, then $\varrho(z,x^0)$ (75) is always at least equal to 1. Next, indefinite integral of function (76) is proportional to function $\frac{1}{z+x^0+D}$ that is infinite at $z + x^0 + D = 0$.

On account of the above we conclude, first, on a restricted interval that do not contain point where $z + x^0 + D = 0$ integral (75) openly depends on time. Second, on unrestricted interval the integral is infinite.

Moreover, as E and p are not constants but functions, the states $\Psi_{+\frac{1}{2}}(z, x^0)$ (58) and $\Psi_{-\frac{1}{2}}(z, x^0)$ (62) can not even be normalized in the sense that $\Psi \dagger \Psi$=1 because the wave functions can not be divided by any combination of E and p that is just opposite the case of free electron. Fortunately, one can them normalize (only) at one point of space-time and it will make possible below to calculate the ratios of transmitted and reflected currents through a uniform electric field barrier.

According to the postulates of wave mechanics, probability can not openly depends on time or be infinite. In our opinion, $\varrho(z, x^0)$ (74) can not be interpreted as probability density and wave functions (58) and (62) do not belong to a Hilbert space.

As regards the general form of solutions (58) and (62), they could be represented schematically as

$$\Psi_{\pm\frac{1}{2}}(z, x^0) = C_1 u_p(z, x^0) e^{\frac{i}{\hbar} S_\pm(z, x^0)}, \tag{77}$$

where C_1 is a constant, bispinor $u_p(z, x^0)$ formally takes the same form as in the case of the free-electron (see Appendix A) but constant values E and p have been replaced with functions $E_\pm(z, x^0)$ (53) or (63) and $p_\pm(z, x^0)$ (55) or (64). In turn, function $S_\pm(z, x^0)$ very formally could be called an action.

5. Quasi-classical interpretation of the non-stationary solution of the Dirac equation

Take a look at the resulting formulas (53,55) and (63,64) for energy and momentum. Variables of position and of time do not depend on themselves. One can see that for each pair of variables (x, ct) we obtain a generally different values of variables E and p. An electron can take every position along the z axis at every time ct (for the convenience we do not distinguish t from ct). This is completely different than in classical mechanics, where a particle always travels along a curve and at a given moment of time ct the particle can take only one position z.

From [4,5] you can find out that an elementary particle when passes through accelerating tube of electrostatic accelerator behaves almost like classical point-like object. This brings us to the conclusion that quantum mechanics can not properly describe the motion of charged particles in a uniform electric field.

Therefore, we are going to deviate from quantum theory and into formulae for energy and momentum will substitute values of variables z and ct resulting from relativistic Newton's second law of motion. In other words, we limit available values of position and time to a subset of the (z,ct)-plane.

According to the relativistic classical mechanics [13], the second Newton's law of motion when applied to a charged particle moving in a uniform electric field is

$$\frac{d}{dt}\left(\frac{mv}{\sqrt{1-\frac{v^2}{c^2}}}\right) = q\varepsilon. \tag{78}$$

Let us remind that we have solved the Dirac equation with a potential $A_0 = \varepsilon z$, where the electric field is -$\varepsilon\mathbf{k}$ and the electron charge is -e. The electric force acting on the electron is equal to $e\varepsilon\mathbf{k}$ and the electric charge moves in the positive direction of the z-axis .

Therefore, equation (78) correctly describes the motion of our electron, if $q = +|e|$. Integrating this equation we obtain that

$$\frac{mv}{\sqrt{1-\frac{v^2}{c^2}}} = p_0 + q\varepsilon t, \tag{79}$$

where p_0 is the linear momentum of the electron at the moment of time t=*0*. From equation (79), in turn one calculates that electron velocity is given as

$$v = \pm\frac{c(\frac{p_0}{q\varepsilon}+t)}{\sqrt{(\frac{p_0}{q\varepsilon}+t)^2+\frac{m^2c^2}{q^2\varepsilon^2}}}. \tag{80}$$

Since $v = \frac{dz}{dt}$ the above equation can be considered as a differential equation for z. Integrating it by means of a substitution we conclude that the position z and the time ct are related to each other by the following equation

$$z = C \pm \sqrt{(\gamma+ct)^2+\delta^2}. \tag{81}$$

Here, C is a constant, $\delta = \frac{mc^2}{e\varepsilon}$, $\gamma = \frac{p_0 c}{e\varepsilon}$ and we took $q = e$ and $e > 0$.

We set about analysing the motion of spin-up electron and ask question whether we can adjust the constant D in formulae (53) and (55) to the constant C in the formula (81) in order that the formulae (53) and (55) could correctly describe the motion of the electron?

For simplicity let $p_0 = 0$. We assume that at time $ct = 0$ the electron is located at $z = 0$ and then moves in the positive direction of the z-axis. Then must be

$$z = -\delta + \sqrt{t^2c^2+\delta^2}. \tag{82}$$

Similarly, so that $E_{+\frac{1}{2}}$ be equal to mc^2 and $p_{+\frac{1}{2}}$ to *0* at point $ct = 0$ and $z = 0$ constant D must be equal to δ.

Energy $E_{+\frac{1}{2}}$ can be expressed as a function of z alone. In order to do that we have to solve equation (82) for variable ct and obtain

$$tc = \pm\sqrt{(z+\delta)^2 - \delta^2}. \tag{83}$$

Since the electron 'moves' along ct axis only in the positive direction we choose sign '+' in (83) and insert it into (53). After some operations applied to (53) we finally obtain desirable result

$$E_{+\frac{1}{2}} = mc^2 + e\varepsilon z. \tag{84}$$

On the other hand, momentum $p_{+\frac{1}{2}}$ can be easily expressed as a function of t alone. We insert (82) into (55) and $D = \delta$, after some operations we get

$$p_{+} = e\varepsilon t. \tag{85}$$

As regards energy $E_{-\frac{1}{2}}$ (63) and momentum $p_{-\frac{1}{2}}$ (64) the same calculations can be made. Therefore, equations (84) and (85) also hold for electron spin-down wave function.

Since application of formulae (82) and (83) gave good results we are going to investigate the case thoroughly.

Let us remind the well-known fact about Schrödinger equation. If its solution can be written as

$$\Psi = ae^{(i/\hbar)S}, \tag{86}$$

then the function S satisfies the Hamilton-Jacobi equation if we neglect the term containing $\hbar^2$ [18,19]. S is the action of a particle. And what about the Dirac equation in the uniform electric field?

At first let us consider the relativistic Hamilton-Jacobi equation. If electron motion is described by equations (26,27) then its Hamilton's function is

$$H = c\sqrt{p_z^2 + m^2c^2} - e\varepsilon z. \tag{87}$$

As Hamiltonian (87) does not involve time explicitly, then the action can be given as

$$S = S_0(z) - ht, \tag{88}$$

where h is an arbitrary constant and the Hamilton-Jacobi equation takes the following form [20,21]

$$H(z, \frac{\partial S_0}{\partial z}) - h = 0. \tag{89}$$

Since momenta p_x and p_y are equal to 0 one can write $S_0 = S_z$, of course.

In order to set the constant h we will use the above assumption that at time $ct = 0$ the electron is located at $z = 0$ where it is temporarily at rest, its momentum is equal to 0 and potential energy also equal to 0, and next it moves in the positive direction of the z-axis. Then from (87,89) appears that

$$h = mc^2, \tag{90}$$

and finally the Hamilton-Jacobi equation we have to integrate is

$$c\sqrt{(\frac{\partial S_z(z)}{\partial z})^2 + m^2c^2} - e\varepsilon z - mc^2 = 0. \tag{91}$$

Integrating (91) and inserting resulting S_z into (88) we obtain classical action of an electron in a uniform electric field in the form

$$S_{cl} = C - \frac{m^2c^3}{2e\varepsilon} ln\,|z+\delta+\sqrt{(z+\delta)^2-\delta^2}| + \frac{e\varepsilon}{2c}(z+\delta)\sqrt{(z+\delta)^2-\delta^2} - mc^2t, \tag{92}$$

where $\delta = \frac{mc^2}{e\varepsilon}$ as above.

In turn, according to (77,86), the 'quantum' candidate for spin-up electron action in this field is

$$S_{qu} = \hbar[-\frac{\omega^2}{2} ln \mid 2(z+x^0)\sqrt{\alpha} + C \mid + \alpha(z^2 + 2zx^0 - (x^0)^2) + C(z-x^0)\sqrt{\alpha}\,], \tag{93}$$

an expression contained in (58). One can easily see that (93) is not well-defined. In order to remedy this we will again use the relation (83) with sign '+' and the result is

$$S_{qu}(z = z(t)) = -\frac{m^2c^3}{2e\varepsilon} ln\sqrt{\frac{e\varepsilon}{\hbar c}} - \frac{m^2c^3}{2e\varepsilon} ln\,|z+\delta+\sqrt{(z+\delta)^2-\delta^2}| + \\ + \frac{e\varepsilon}{2c}(z+\delta)\sqrt{(z+\delta)^2-\delta^2} - mc\sqrt{(z+\delta)^2-\delta^2}. \tag{94}$$

Since classical mechanics does not know Planck's constant $\hbar$ we have to set C=0 in (92) and to drag constant $-\frac{m^2c^3}{2e\varepsilon} ln\sqrt{\frac{e\varepsilon}{\hbar c}}$ in (94) into constant C_1 in (58). The last term in (94) with formula (83) can be written as $-mc^2t$ and at last we have

$$S_{cl} = S_{qu}(z = z(t)). \tag{95}$$

Due to the similarity of wave functions (77) to the free electron solutions it is worth asking whether the general form of the solution (77) has something to do with used in quantum mechanics the quasi-classical wave function?

Let us recall that the classical action for a motion of free electron has the form

$$S = \mathbf{p} \cdot \mathbf{r} - Et + C, \tag{96}$$

and it perfectly agrees with the formulas (.116) and (.117). Through the action of energy and momentum operators on (.116) and (.117) we obtain its correct eigenvalues.

If instead of function $S_{\pm}(z, x^0)$ we put into (77) the classical action in a uniform electric field S_{cl} given by equation (92), we will get that the eigenvalue of energy operator q^0 (51) acting on (77) is equal to that of formula (84) and the eigenvalue of momentum operator q^3 (51) also acting on it is

$$\frac{e\varepsilon}{c}\sqrt{(z+\delta)^2 - \delta^2} \tag{97}$$

that is equal to (85) through (83). In conclusion we can say that

$$\Psi_{\pm\frac{1}{2},quasi-classical} = \Psi_{\pm\frac{1}{2}}(z, x^0)|_{z=z(x^0)}, \tag{98}$$

where $\Psi_{\pm\frac{1}{2}}(z, x^0)$ is given by (77) and $|_{z=z(x^0)}$ means that electron moves along classical trajectory $z = z(x^0)$.

6. The probability of transmitting electrons through a uniform electric field barrier

The goal of the following investigation is to calculate the transmission coefficient for a potential of the form

$$A_0(z) = 0,\ z < 0, \tag{99}$$

call it region I and

$$A_0(z) = \varepsilon z,\ 0 \leqslant z < +\infty, \tag{100}$$

call it region II, where as previously ε is a positive constant. The electron approaches from a region of negative z and is reflected or transmitted by the barrier of linearly increasing potential.

Due to the different forms of the potential the two regions must be treated separately [14]. In region I (for $z < 0$) the spin-up solution of the Dirac equation is (Appendix A)

$$\Psi_{I,+\frac{1}{2}}(z,t) = a\begin{pmatrix} 1 \\ 0 \\ \frac{E-pc}{mc^2} \\ 0 \end{pmatrix} e^{\frac{i}{\hbar}(pz-Et)} + b\begin{pmatrix} 1 \\ 0 \\ \frac{E+pc}{mc^2} \\ 0 \end{pmatrix} e^{\frac{i}{\hbar}(-pz-Et)}, \tag{101}$$

where the second part of the wave function describes the reflected wave. In region II (of constant electric field) the spin-up electron is represented by the following wave function

$$\Psi_{II,+\frac{1}{2}}(z,x^0) = d \begin{pmatrix} 1 \\ 0 \\ \frac{E-pc}{mc^2} \\ 0 \end{pmatrix} e^{i[-\frac{\omega^2}{2} ln|2(z+x^0)\sqrt{\alpha}+C|+\alpha(z^2+2zx^0-(x^0)^2)+C(z-x^0)\sqrt{\alpha}]}. \tag{102}$$

The Dirac equation contains only first-order derivatives and therefore we need to impose on the solutions (101) and (102) only one requirement that $\Psi_{+\frac{1}{2}}(z,t)$ be continous at $z=0$ and $t=0$.

Let electron energy and momentum before the region of non-zero field are arbitrary. In order not to lose generality of our calculations we will take energy and momentum of the electron in the electric field as E' and p', i.e. they will be different from those of the free electron and the constant in (102) must be marked as C' (and D' respectively).

When $z=0$ and $t=0$ the values of the exponents of solution (101) are equal to 0 but the same of (102) is equal to

$$-i\frac{\omega^2}{2} ln|C'|, \tag{103}$$

where due to the previous adjustment of constant D (50) the constant C' is equal to

$$C' = 2D'\sqrt{\alpha}. \tag{104}$$

Requiring that the wave function must be continous at $z=0$ and $t=0$ leads to the relations

$$\begin{aligned} a+b &= d \cdot e^{-i\frac{\omega^2}{2} ln|C'|}, \\ \frac{a(E-pc)}{mc^2} + \frac{b(E+pc)}{mc^2} &= \frac{d(E'-p'c)}{mc^2} \cdot e^{-i\frac{\omega^2}{2} ln|C'|}. \end{aligned} \tag{105}$$

From these two equations we may find the ratios $\frac{b}{a}$ and $\frac{d}{a}$ which take the forms

$$\begin{aligned} \frac{d}{a} &= \frac{2pc}{(p'+p)c-(E'-E)} e^{+i\frac{\omega^2}{2} ln|C'|}, \\ \frac{b}{a} &= \frac{2pc}{(p'+p)c-(E'-E)} - 1. \end{aligned} \tag{106}$$

By proper adjustment of constant D (in fact D') in formulae (53) and (55) we can make energy *E'* and momentum *p'* of the electron in the electric field at z = 0 and t = 0 equal to energy E and momentum p of free electron.

If *E'=E* and *p'=p*, then the ratio of the transmitted current to the incident is $\frac{|d|^2}{|a|^2}$, which is equal to 1 and the ratio of reflected current to the incident is $\frac{|b|^2}{|a|^2}$, which is equal to 0. Thus all of the incident current is transmitted.

7. Conclusions

The need to find solutions to the Dirac equation in a uniform electric field appeared in connection with the Bohr's discussion with Heisenberg and Sommerfeld on the Klein paradox [15]. Sauter [1] and Plesset [2] sought stationary solutions, but did not notice that in this field the Dirac equation has no such solutions.

Myers [3] was the first to obtain a non-stationary solution in that field. He introduced to the equation a trial function given by formula (4) and formulated the quantum mechanical expression for momentum using the fact that in the classical mechanics the x-component of mechanical momentum is

$$p_x = p_{0x} + \epsilon E t, \tag{107}$$

where ϵ is electron charge and E electrostatic field strength. Myers did not explain why he had looked for non-stationary solution. In our opinion, Myers' solution is not exact.

Above we have showed that the wave function of an electron moving in a uniform electric field can be only non-stationary and presented its another (and exact) two forms $\Psi_{+\frac{1}{2}}(z,x^0)$ Eq.(58) and $\Psi_{-\frac{1}{2}}(z,x^0)$ Eq.(62). They are eigenfunctions of energy and momentum operators. However, its eigenvalues are not well-defined observables because they simultaneously depend on position z and time t.

In order to compare Myers' solution to ours we will use the language of the Ritz variational method [12]. Although the method is for aproximate determination of energy levels of the discrete spectrum, however, in our opinion, its terminology here will be very useful. Myers' and our solutions belong to certain function spaces. In order to find a solution Myers made many more assumptions as for the form of it than we did. In particular, he chose a specific form of particle momentum but we did not. In other words, the space of his solution is more restricted than of ours. The same our solution is more general and broad.

There is another very important fact about Myers' wave function. As formula (107) comes from classical mechanics it contains hidden assumption that an electron in a uniform electric field moves along classical trajectory, i.e. its position is a function of time. Since Myers used (107) before obtaining his solution he violated the Dirac equation where any relations between variables x, y, z and t are not allowed. Equation (107) is only classical surogate used in quantum mechanics. In our opinion Myers' solution is only semiclassical by contrast with ours that is pure quantum mechanical.

To solve a nonstationary problem in quantum mechanics is not easy task. In the relativistic one there is only one more system for which a nonstationary solution can be found exactly. It is the case of an electron in the field of an electromagnetic plane wave solved by Volkov [6,17]. Its kinetic momentum also depends on position and time but only through the vector potential of the plane wave that is why its time-average value is well defined.

In [20] Rubinowicz and Królikowski noted that if the time-dependence of the wave-function can not be separated from the spatial dependence then on the whole one can not attribute definite values of energy to the system. So the fact that energy and momentum operators eigenvalues (53), (55), (63), (64) are not well-defined is only general property of non-stationary solutions of quantum equations.

Taking the above conclusions into account one can try to formulate the following hypothesis. Since the quantum mechanics is not able to give definite values of energy and momentum of an electron, it is worth to use for that purpose any other law of physics.

In section 'Quasi-classical interpretation of the non-stationary solution of the Dirac equation' we have applied with this end in view Newton's second law. It gave correct values of electron energy and momentum and also correct classical action. That is the source of unexpected conclusion that in the uniform electric field an electron displays not only quantum features but also the classical ones. The name 'quasi-classical' has been attributed to the Section 5 of caution. In our opinion, in that field an electron really moves along classical trajectory.

The natural question appears what experimental application the new solution of the Dirac equation could have? We will briefly explain.
We also tried to solve the same way the Klein-Gordon equation describing bosons that has no spin. To obtain the equation in terms of variables x (28) and y (29) is sufficient to remove term '$-i$' from equation (33). The term describes the interaction between electron spin and the uniform electric field.

The suitable trial function is

$$\varphi^1 = exp[i\{(x+y)^2/2 - xy + bf(x,y)\}]. \tag{108}$$

Inserting formula (108) into the Klein-Gordon equation results in the following set of equations

$$\frac{\partial^2 f}{\partial x \partial y} = 0, \tag{109}$$

$(\frac{\partial f}{\partial x} = \frac{dg}{dx}, \frac{\partial f}{\partial y} = \frac{dh}{dy})$ and

$$(x - b\frac{dh}{dy})(2x + y + b\frac{dg}{dx}) - \omega^2 = 0. \tag{110}$$

These two equations are contradictory to each other. One could integrate the Klein-Gordon equation the above way if it contained a term similar to the term that describes the interaction between electron spin and an electric field.

The combination of the above conclusion and the calculations we have made in section 5 indicate that interaction between the electron spin and the uniform electric field determines the fact that the motion of the electron in that field takes place in accordance with the Newton's second law.

Hence, in our opinion, it would be desirable to make sure that movement of charged spinless bosons in that electric field also takes place in accordance with the Newton's second law. We have doubts about that.

Appendix A In order to obtain free solutions of the Dirac equation in the spinor representation we return to equations (26) and (27), take into account that relativistic hamiltonian for a free particle does not contain electric charge ($e = 0$) and the equations become

$$(\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial (x^0)^2} - \frac{m^2 c^2}{\hbar^2})\varphi = 0, \tag{.111}$$

and

$$\chi = \frac{i\hbar}{mc}(\frac{\partial}{\partial x^0} + \sigma_z \frac{\partial}{\partial z})\varphi. \tag{.112}$$

As a trial function we take

$$\varphi^1 = Ce^{\frac{i}{\hbar}(pz - Et)} \tag{.113}$$

and substitute it into (.111). Thus we obtain the familiar relation

$$E^2 = p^2 c^2 + m^2 c^4. \tag{.114}$$

Substitution of (.113) into (.112) gives

$$\chi^1 = \frac{E - pc}{mc^2}\varphi^1, \tag{.115}$$

and finally is

$$\Psi_{free,+\frac{1}{2}}(z,t) = C \begin{pmatrix} 1 \\ 0 \\ \frac{E-pc}{mc^2} \\ 0 \end{pmatrix} e^{\frac{i}{\hbar}(pz-Et)}. \tag{.116}$$

Similarly, in the case of spin-down function φ^2 as a trial function we again take (.113) and obtain

$$\Psi_{free,-\frac{1}{2}}(z,t) = C \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{E+pc}{mc^2} \end{pmatrix} e^{\frac{i}{\hbar}(pz-Et)}. \tag{.117}$$

Free electron solutions (.116) and (.117) can be represented schematically as

$$\Psi_{free,\pm\frac{1}{2}}(z,t) = C u_p e^{\frac{i}{\hbar}(pz-Et)}, \tag{.118}$$

where C is a constant and u_p is a four-component bispinor independent of $\boldsymbol{r}$ and t contained in (.116) or (.117).

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